Binary Automatic Control Systems

S.V. EMELYANOV

Advances in Science and Technology in the USSR

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BINARY AUTOMATIC CONTROL SYSTEMS
by S.V. Emelyanov

The book outlines a new approach to the synthesis of nonlinear dynamic systems – a key problem in control engineering. The suggested design formalism is based on the "binary" principle which is essentially a dual treatment of signals in the dynamic system – as a phase variable in one circumstance and as a transformation operator in another. This principle opens up the possibility of handling control laws as ordinary phase variables and invokes feedback to generate such laws. The feedback mechanism in this case is exploited in a nonorthodox manner, leading to the occurrence of new types of feedback.

The efficiency of the new approach is demonstrated by constructing control systems which stabilize motion of nonlinear dynamic plants with ill-defined dynamics. The stabilization is attained with continuous control and bounded gains.

Apart from solving the problems of ill-defined plant control, the binary principle is used to advantage for the development of nonlinear filters separating close frequencies, for multiple differentiating noisy signals, and for control of multiconnected systems.

The value of the binary principle is not confined to the control field; rather it may be employed in studies of complicated dynamic systems in biology, biomechanics, social sciences, robotics, and biotech engineering, to name a few of the most promising applications.

About the Author

S.V. Emelyanov, Fellow of the Academy of Sciences of the USSR, has been engaged in solving many diverse problems of control. He is renowned for his pioneering efforts in the development of variable structure systems. The idea to incorporate unstable structures and sliding modes into the closed-loop control system for the purpose of system's stabilization was also devised by the author of this book.

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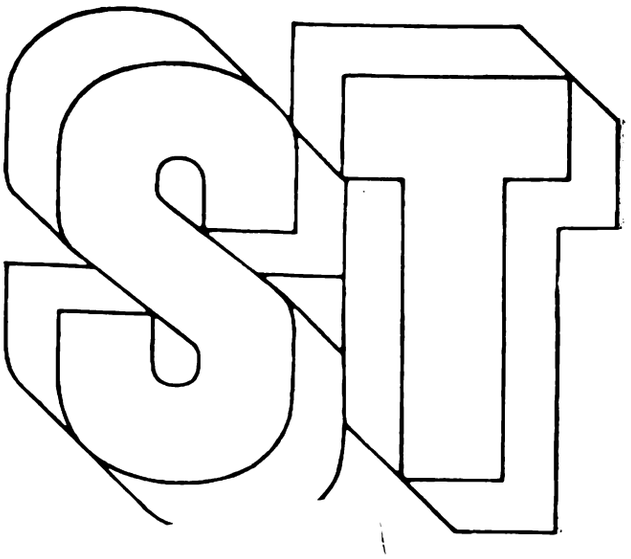

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CONTENTS

	Introduction	7
I	**BINARY SYSTEMS OF AUTOMATIC CONTROL. MOTIVATION, DEFINITIONS AND CONCEPTUALIZATION**	10
1	**Basic Principles and Definitions**	10
1.1	Functional Diagrams of Control Systems in Classical Control Theory Setting	10
1.2	Block Diagrams of Adaptive Control Systems	12
1.3	The Concept of Operator-Variable and the Binary Principle	14
1.4	Generalized Elements of Binary Dynamic Systems	16
1.5	New Types of Feedback	17
2	**The Principles of Control Under Uncertainty**	18
2.1	Three Principles of Control for Solving Control Problems	18
2.2	Methods of Control for Ill-Defined Dynamic Systems	23
3	**Generalized Block Diagrams of Binary Control Systems**	28
3.1	Constructing Binary Control Systems	28
3.2	Structures of Binary Control Systems with Coordinate-Operator Feedback	29
3.3	Structures of Binary Control Systems with Coordinate-Operator and Operator Feedback	32
3.4	Structures of Binary Control Systems with Coordinate-Operator, Operator, and Operator-Coordinate Feedback	34
II	**FREE MOTION CONTROL**	36
4	**Definitions and Nomenclature**	36
5	**Time-Invariant Linear Coordinate Feedback**	41
5.1	Finite Gain	41
5.2	High Feedback Gain	42
5.3	Constrained Control Signal	44
5.4	Imperfections	46
5.5	Linear Zone	49
5.6	Coordinate Feedback Under Inaccurate Information	51
5.7	Inaccuracies in Model Approximation	53
6	**Coordinate-Operator Feedback**	55
6.1	Proportional Coordinate-Operator Feedback	59
6.2	Proportional Coordinate-Operator Feedback with Magnitude Bounded Output	61
6.3	Nonlinear Time-Invariant Coordinate-Operator Feedback	67
6.4	Bang-Bang Coordinate-Operator Feedback	75
6.5	Bang-Bang Coordinate-Operator Feedback with Constraints and Dynamic Nonlinearities	79
6.6	Integral Coordinate-Operator Feedback	89
6.7	Integral Coordinate-Operator Feedback with a Constant Integration Rate	92
6.8	Inertial Coordinate-Operator Feedback	110

6.9	Integral Coordinate-Operator Feedback with Variable Integration Rate	118
6.10	Continuous Inertial Coordinate-Operator Feedback	124
6.11	$S_\mu^r(2)$ Systems Under Integral Coordinate-Operator Feedback Law	128
6.12	Application of $\mathcal{A}_\mu$ Algorithms with Inertial Coordinate-Operator Feedback to S System Control	136
7	**Operator Feedback**	**154**
7.1	$S_{\mu\rho}$ (1) Systems with Integral Coordinate-Operator Feedback	160
7.2	S_μ and $S_{\mu\rho}$ Systems Under Imperfect Information	168
7.3	$S_{\mu\rho}$ (1) Systems with Inertial Coordinate-Operator Feedback	189
7.4	Quasicontinuous Control Algorithm	192
7.5	$S_{\mu\rho}$ (2) Systems with Integral Coordinate-Operator Feedback	206
7.6	S System Control with $\mathcal{A}_{\mu\rho}$ Algorithms	214
8	**Operator-Coordinate Feedback**	**221**
8.1	OCFB Generation Concepts	224
8.2	$S_{\mu\nu}$ (1) Systems with Proportional and Integral Coordinate-Operator Feedback	236
8.3	$S_{\mu\rho\nu}$ (1) Systems with Proportional Operator-Coordinate and Integral Coordinate-Operator Feedback	247
8.4	$S_{\mu\nu}$ (1) Systems with Integral Laws of Coordinate-Operator and Operator-Coordinate Feedback	253
8.5	$S_{\mu\rho\nu}$ (1) Systems with Integral Laws of Coordinate-Operator and Operator-Coordinate Feedback	258
8.6	$S_{\mu\rho\nu}$ (1) Systems with Inertial Laws of Coordinate-Operator and Operator-Coordinate Feedback	264

References

Index

INTRODUCTION

Modern industrial technology involving sophisticated machinery, flexible automatic production lines, and wide-scale robotization presents system technologists and industrial managers with a number of new problems. They can be solved only on the condition that the functional capabilities of available automatic control systems be further expanded and the systems themselves acquire the quality of rapid adaptation to varying environmental and operational conditions including those brought about by the properties of the plant being controlled. Wide-scale implementation of such control systems would alleviate many labor consuming problems of system design, adjustment, and operation.

At present the development of automatic control systems is a complicated multistage process involving the expert skill of analysts at each stage. As a rule, a development begins with an analysis of the object of automation. A mathematical model of the process is built on the basis of the pertinent laws of natural sciences. This evolves as a manifold of cause-effect relationships. The principal difficulties facing the analyst at this stage consist in deciding on to what degree of detail the model should be elaborated. No ready recipes exist for selection of optimum model size. One may devote much time to problem evaluation and arrive at a very sophisticated model reflecting many subtle features of the plant. Such a model will considerably complicate the subsequent stages of control system development while the contribution of the fine features to the final result may prove insignificant. Conversely, a relatively simple mathematical model may be rapidly built and bring about good results.

Choosing a model, therefore, is a nonformal creative stage, and what comes out of it must be consistent with the overall objective. The model size should take into account the technical means available in the implementation of the control system concerned. In any case, the model mirrors to some approximation the behavior of the physical system. It is generated as a set of equations and relations between the physical variables (or their functions) whose time variations are essential for solving the control problem at hand. Other variables either remain unrepresented in the mathematical description of the controlled system or enter this description as parameters. The latter not only define the nature and ranges of parameter variations with time but also give the degree of uncertainty under which the model will have to be handled as the variation laws for these variables are not known exactly. This implies that the analyst working out control laws has to tackle uncertain (ill-defined) dynamic systems.

The second stage of development of such a control system is therefore concerned with the elimination of this uncertainty. Here the designer strives to identify the model parameters, establish their time variation laws or if impossible estimate the variation ranges for these parameters. For many important applications the identification problem has to be solved not only at the design phase but in system operation as well, either intermittently or consistently (on line). The identification problem is critical and complicated. Its solution is pivotal for system performance as a whole.

The next stage of system development concerns itself with the choice of a feedback law. First, the general form of control law is determined from the mathematical model and the objectives of control, then the solutions to the identification problem are incorporated to compute the parameters of the chosen law. In some applications, now rather rare, the solution of problems occurring at this stage may turn out relatively simple. Other situations necessitate solving auxiliary problems involving large amount of computational work, and even additional investigations. The procedure can be improved to a certain degree by invoking computer aided design (CAD) systems. It is quite obvious that a nonstationary plant requires that the control law parameters be corrected consistently, and this correction be set on line by skilled personnel.

Once the problems of the previous stage have been solved, the general structure of automatic control system becomes clear enough to be realized by technical means. The latter naturally can perform the necessary functions to some accuracy, thus introducing additional constraints and interferences into system performance. Allowance for these constraints is not always possible at the preceding stages hence a simulation experiment is needed. This experiment is carried out with all, or a considerable part, of equipment being the same as that expected to be in the actual system, while the plant and some measuring and actuating devices are represented by simulation models. The objective of this experiment is to test the designed system for compliance with the performance specifications and, if necessary, to introduce corrections into the settings of controller's parameters.

The final stage of development is devoted to performance tests of the control system. Some final adjustments are introduced into the control system on site as needed occasionally by actual operation and effected by the maintenance personnel.

The list of problems as given above is typical of control system development. The solution to these problems requires the efforts of many skilled designers. Under the conditions of large-scale automation the need for such experts increases sharply and may be a stumbling block on the way to extensive automation in the industrial and social milieus.

A possible solution to this problem may be the creation of a control theory which would be capable of handling ill-defined plants, accounting for actual constraints on the variables, parameters, and controls intrinsic to the plant and technical means, excluding the need for exact identification of model parameters and ensuring efficient plant operation at arbitrary variation of circuit parameters over wide ranges. The respective control algorithms, of course, may become rather sophisticated and require considerable computer capacities for their implementation. However the progress in computing achieved in the last decades has provided a strong foundation for practical implementation of the new approach to automation. The reliable, cheap, and efficient microprocessor hardware has changed drastically the idea of the technical means of automation and has eliminated many traditional limitations associated with the hardships of implementation and the reliability of controllers.

All the aforementioned ideas and facts indicate that control system designers are faced with new problems and at a new level of possibilities in the implementation of control algorithms. New methods, principles and procedures of control system development are therefore highly desirable. A new theory of automatic control outlined in this volume is an answer to this appeal. It is devised to tackle the problems of automation outlined in the preceding paragraphs. More specifically it develops new methods of analysis and synthesis for essentially nonlinear systems. The theory is advantageous in that it reduces the amount of necessary prior data and is content with variation ranges instead of the exact characteristics of the plant.

The theory yields a final set of control algorithms each of which is efficient in the control of a wide class of plants. A simple recursive rule of control system buildup is another feature of the theory which in fact is a natural extension of classical control theory. Experience with simulation and application of control systems developed in this way has verified that they can be implemented easily and most efficiently on the basis of computer logic, specifically microprocessor technology.

I Binary Systems of Automatic Control: Motivation, Definitions and Conceptualization

This part outlines the basic principles and definitions of binary dynamic systems. It describes how the error-closure control principle and binary system concepts are used in the development of automatic control systems for ill-defined dynamic processes or plants operated under uncertainty. The principal properties and features of binary control systems are emphasized and compared with traditional control systems devised to solve the same control problem.

1 Basic Principles and Definitions

1.1 Functional Diagrams of Control Systems in Classical Control Theory Setting

Functional block diagrams are widely used in different disciplines to convey the principal features of processes and phenomena under study. Block diagrams are convenient means of graphica representation indicating in a straightforward manner the presence and character of interactions between different processes in complicated systems. They are appealing to the analyst's intuition and facilitate the system reappraisal and alteration to change its behavior in the desired direction. The block-diagram representation has proved efficient in solving many a problem including those of automatic control for systems with lumped [1, 2, 4-6] and distributed [1, 7-10] parameters.

In classical control theory, functional block diagrams a variable, or coordinate, is represented by an arrow (Fig. 1.1*a*), while a dynamic element, or operator, which defines a transformation of the input into output variables, is represented by a block as in Fig. 1.1*b*. Each block diagram picturing a control system contains a certain number of operators and connecting coordinate paths between them.

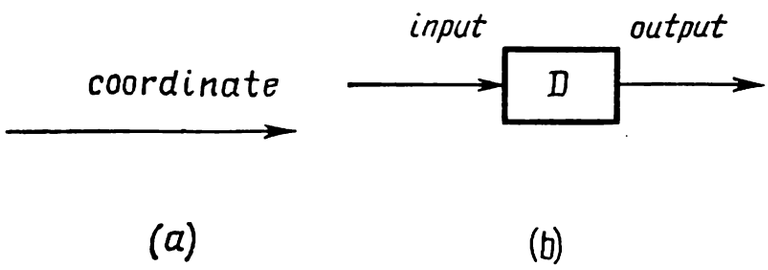


Figure 1.1

A particular type of block diagram depends upon many circumstances, specifically upon the concept of operator being employed—whether the operator represents an individual component or some subsystem constituted by elementary components. Examples of the latter types are the generalized block diagrams given in Fig. 1.2. These block diagrams have received numerous detailed treatments in texts on classical control theory [1-7] and reflect the three basic principles of control, viz. compensation (*a*), feedback (*b*), and combined control (*c*).

In the block diagrams of Fig. 1.2, $y^r(t)$ is the reference input (or setpoint) which is to be reproduced by the controlled variable $y(t)$, with $x(t)$ being the error signal, and $f(t)$ an exterior disturbance acting on the system. The black sector in the summing point shows that the incoming signal arrives with an inverted phase, i.e. has a minus sign.

In classical control theory, the major problem is to appropriately choose operator D_2 representing the controller, time invariant

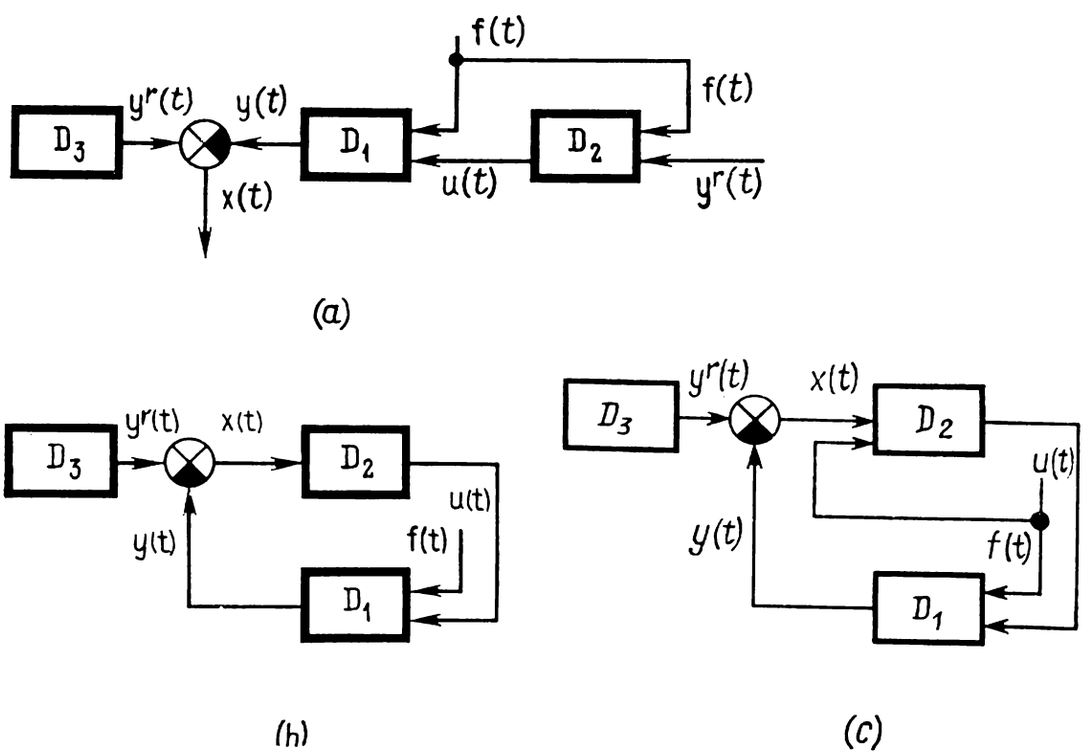


Figure 1.2

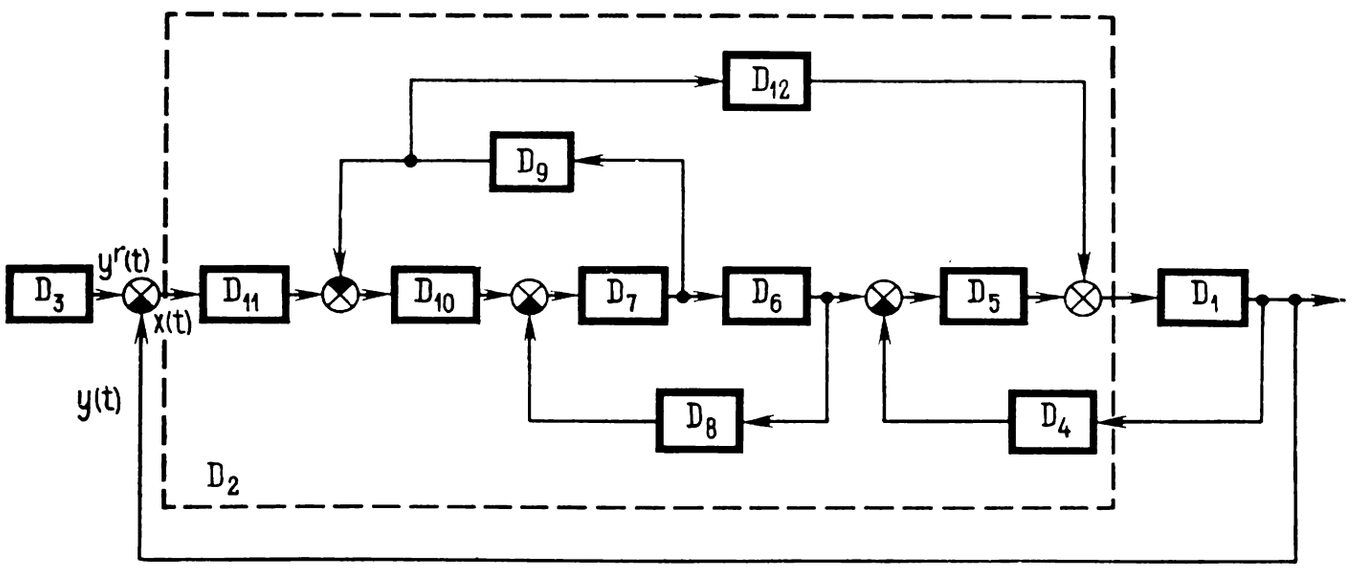


Figure 1.3

as a rule. There is a variety of techniques enabling this choice, dissimilar in many aspects and depending on many factors, specifically on the information that can be employed in the controller to form the control, or manipulated, variable for the controlled plant. Normally, operator D_2 is constituted by elementary components (operators), representing different correcting elements, filters, etc. linked in some arrangement. This arrangement is reflected by the block diagram schematizing the network of operators and different (feedforward, feedback, positive, negative) coordinate paths. An example of such a block diagram, borrowed from Ref. 11, is given in Fig. 1.3.

Classical control theory may be said to deal with systems which can be represented by block diagrams constructed essentially with the use of two basic elements, or concepts, namely, the operator and the variable, or coordinate.

1.2 Block Diagrams of Adaptive Control Systems

The theory of adaptive control has expanded the possibilities of automatic systems. Methodologically this expansion has been due to the acquisition of a new basic element—the variable-parameter operator; its conventional graphical symbol is depicted in Fig. 1.4. The incoming signal $\varepsilon(t)$ represents a function governing the variation of parameters of operator D [11-14].

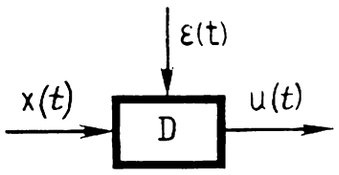
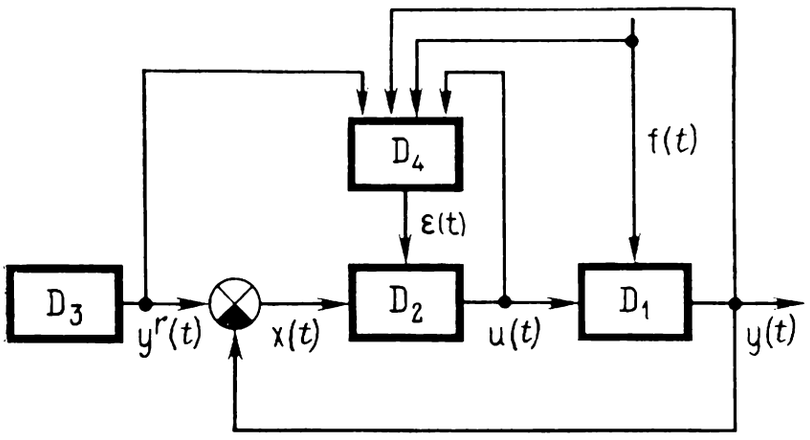
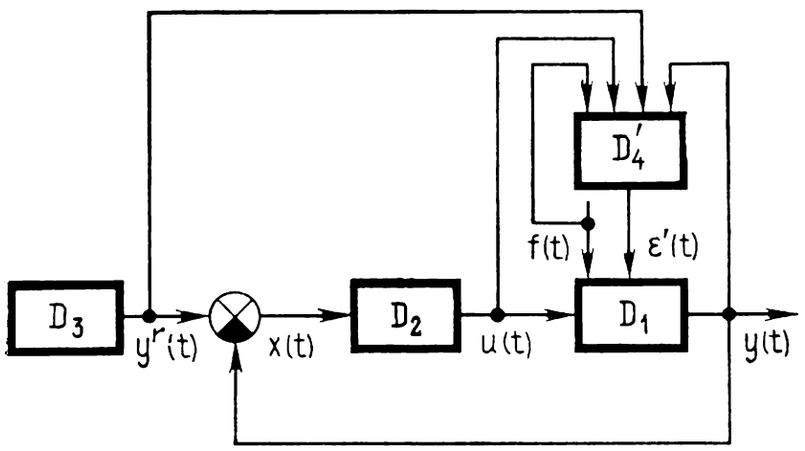


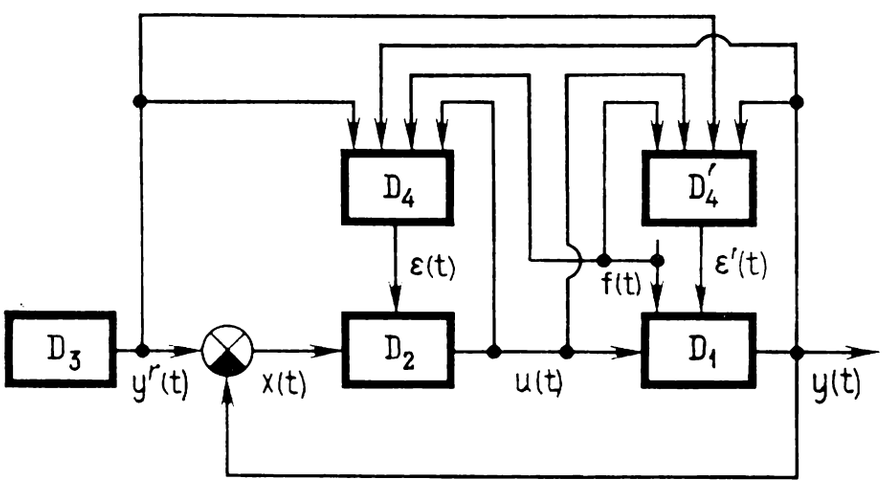
Figure 1.4



(a)



(b)



(c)

Figure 1.5

Numerous reports on the subject have demonstrated that the incorporation of such an element widens the class of feasible controllers

and enables the designer to develop systems capable of efficient control over an entire class of plants.

By way of example, we take up three generalized block diagrams of adaptive control systems which have been studied in sufficient depth. The networks are dissimilar in the way they use the element presented in Fig. 1.4. In the block diagram of Fig. 1.5*a* this element (D_2) represents an algorithm of coordinate control [11, 12]. In the diagram at (*b*) this element (D_1) is used to change some parameters of the plant [13, 15-18]. Finally, the adaptive control network shown at (*c*) realizes both these possibilities [13].

The problem of adaptive control synthesis reduces, therefore, to the choice of a coordinate-control operator D_2 and an operator D_4 (D_4') which defines the adaptation algorithm. A popular approach given the circumstances is as follows. The analyst selects at first an invariable operator D_2 ensuring the desired quality of control at some fixed parameters of the plant corresponding to its most typical operating mode, and then synthesizes an operator D_4 which will correct the parameters of D_2 to bring the actual closed-loop system characteristics in correspondence with the specified values.

To sum up, the adaptive control systems may be said to correspond to block diagrams constructed with the three basic elements, namely, operators, variable parameter operators, and coordinate paths between them [13, 19-23].

1.3 The Concept of Operator-Variable and the Binary Principle

In what follows we develop a new methodological basis for constructing block diagrams with the aim to widen the capabilities of automatic control systems handling dynamic plants under uncertainty. This methodology relies on the concept of operator-variable, or operator-signal. In block diagrams, this variable will be represented by a double-shafted arrow (Fig. 1.6).

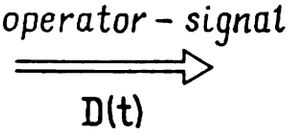
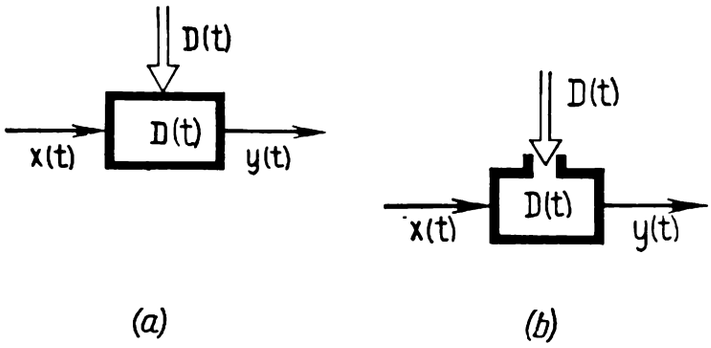


Figure 1.6

The *operator-variable*, or *operator-signal*, formally represents a transformation applied to variables (coordinates). The concept of operator-signal enables us to introduce a new, for control theory, element depicted in Fig. 1.7. This element graphically represents the fact that the transformation of coordinate $x(t)$ into $y(t)$ is governed by an external operator-signal $D(t)$.



(a) (b)

Figure 1.7

It would be proper to note here that $D\ (t)$ may be an ordinary signal which undergoes various transformations, that is, performs as a *coordinate*. This implies, specifically, that such a variable may be included in the set of system variables where each generic element of the set can be either a coordinate or an operator. The particular label is decided by the role the element assumes in a specific transformation.

The need for the new element has arisen from the fact that many problems involve on-line alterations of data processing algorithms. Physically, such an element can be implemented in many ways. For example, the type of transformation may be changed parametrically; the element concerned is then identical with the one employed in adaptive networks (see Fig. 1.4). Another way consists in using a multiplication of signals. The element then assumes the form represented graphically in Fig. 1.8*a*. The operator-signal may also assume a logic form. Figure 1.8*b* illustrates, for example, how the choice of certain $D_1, D_2, \ldots, D_N$ is governed by the operator-signal $D\ (t)$. It will be noted that the logic method of varying the transformation operators is widely used in control systems with variable structure [24, 25].

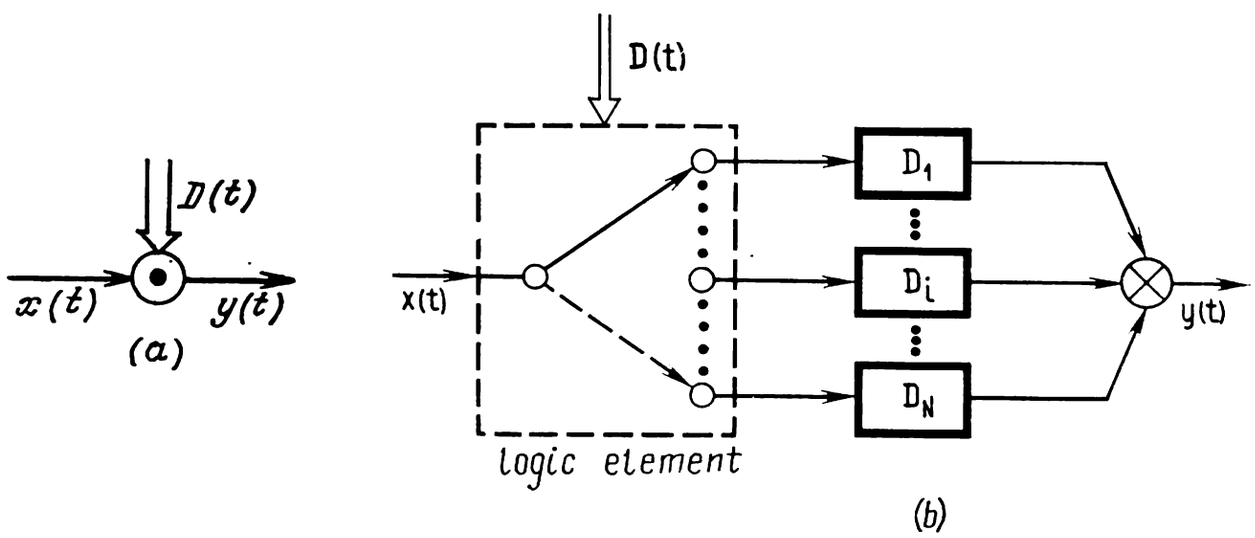


Figure 1.8

In practice, certain signals are responsible for changing the type of transformation, therefore, it would be natural to introduce for them a distinct nomenclature. They will be referred to as *operator-variables*, or *variant-operators*, and denoted by Greek letters shown next to double-shafted arrows (Fig. 1.9). The distinction between

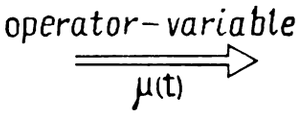


Figure 1.9

variables referred to as *coordinates* and *operator-variables* is merely conditional. For a casual reader unaware of the aforementioned distinction of variables, these are essentially the same—the variables of the nonlinear dynamic system under consideration. The interpretation of these variables is a methodological means whose efficiency in system synthesis and analysis is yet to be demonstrated. In view of the importance of the concepts being introduced for our further consideration, we would like to emphasize once again that there is no formal difference between coordinates and operator-variables. The difference is conditional and associated with the way in which the variable participates in specific local transformations.

We shall call a variable the *coordinate* if it is subjected to some transformation. The same variable will be called the *operator* when it determines the type of transformation performed over a coordinate. This interpretation of state variables of a nonlinear dynamic system will be referred to as the *binary principle*. Dynamic systems built on this principle will, accordingly, be called *binary dynamic systems*.

1.4 Generalized Elements of Binary Dynamic Systems

Here we are content to define the conceptual framework that will be necessary in constructing a theory of binary control systems. Like any variable, the operator-variable may be subjected to a transformation and if the output of such a transformation performs itself as an operator, then this mapping will be referred to as the operator transformation. It is represented in block diagrams as shown in Fig. 1.10*a*. If, on the other hand, the output is to be subjected to another transformation on its downstream way, then the former mapping will be referred to as an *operator-coordinate* transformation and will be schematized as indicated at (*b*). Similar definitions hold for the *coordinate-operator* type (*c*) and *coordinate* type (*d*) transformations. The last one is an ordinary transformation, it coincides

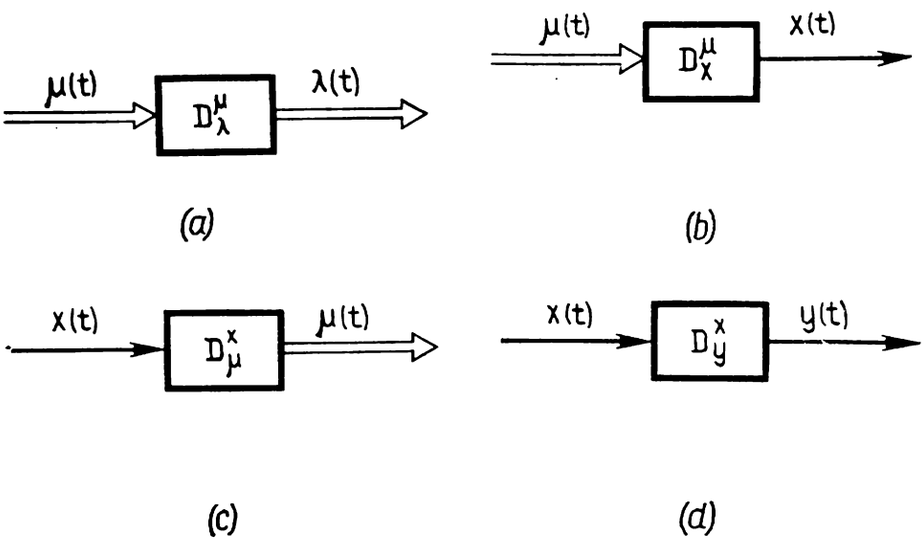


Figure 1.10

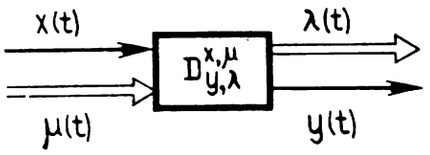


Figure 1.11

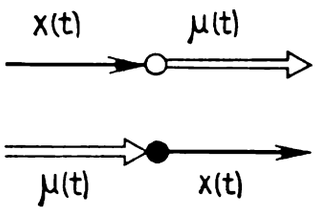


Figure 1.12

with that represented in Fig. 1.1*b*. Mixed type transformations are also possible (Fig. 1.11). Because some variables can take part in several transformations and in different capacity, it would be worth introducing the identity (direct and reverse) transformation of a variable, illustrated graphically in Fig. 1.12.

The concepts and definitions introduced above may be combined with various principles of control theory to yield a great variety of functional block diagrams for binary control systems.

In this work we shall exploit only the principle of error-closing control in our design of binary control systems for essentially non-stationary dynamic processes operating under parametric uncertainty.

1.5 New Types of Feedback

In what follows we shall denote by the letters P, R, and C the operators representing, respectively, the plant to be controlled, reference input, and controller. Specific subscripts will indicate the output variable of these operators. Now, simple closed-loop dynamic systems receive different types of feedback as a result of using the principle of error-closing control and a variety of the aforementioned operators. Figure 1.13*a* shows the control system with *coordinate*

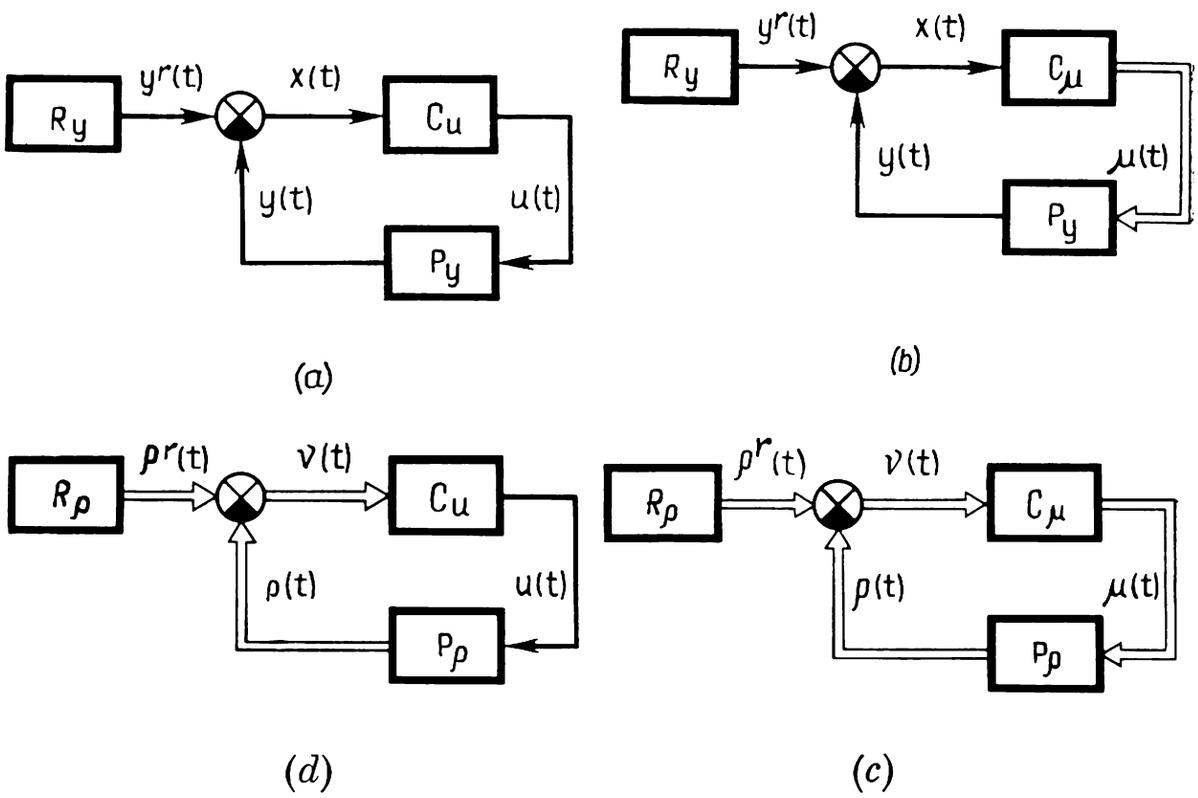


Figure 1.13

feedback (CFB); the diagram at (*b*), the system with *coordinate-operator* feedback (COFB); at (*c*), the system with *operator* feedback (OFB); and at (*d*), the system with *operator-coordinate* feedback (OCFB).

2 The Principles of Control Under Uncertainty

2.1 Three Principles of Control for Solving Control Problems

Consider one of the principal control problems in which the controlled plant, represented in Fig. 2.1 by the operator P_y, is subjected to a control action (manipulated variable) $u\ (t)$ which is to be formed such that the plant output (controlled variables) $y\ (t)$ will reproduce certain time functions $y^r\ (t)$ generated by the reference (setpoint) input R_y. Given initial mismatch is eliminated, the error $x\ (t) = y^r\ (t) - y\ (t)$ is to be varied in transients in some desired manner. For definiteness, here and below we assume that the controlled plant is described by linear ordinary differential equations. As a rule, the solution to this problem is to be achieved

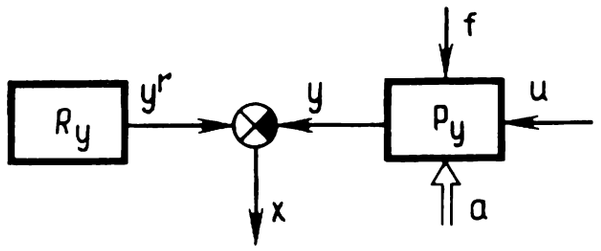


Figure 2.1

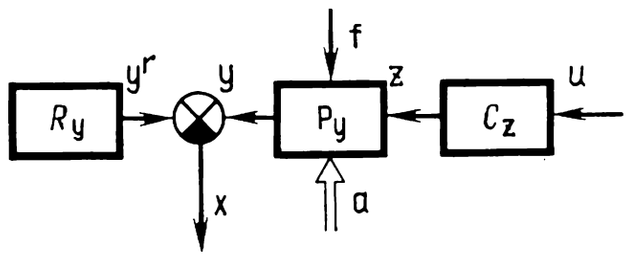


Figure 2.2

for the plant subjected to various disturbances, which can be of operator type (specifically, parametric), $a\ (t)$, and of coordinate type, $f\ (t)$. The former disturbances are represented in Fig. 2.1 by a double-shafted arrow, the latter by a single-line arrow. What this distinction in the exterior disturbances and their designations means will be clear from the following representation of the plant operator P_y:

$$\begin{aligned} y\ (t) &= P_y\ [a\ (t),\ f\ (t),\ u\ (t)] \\ &= P'_y\ [a\ (t)]\ u\ (t) + P''_y\ [a\ (t)]\ f\ (t) \end{aligned} \tag{2.1}$$

The additive part independent of control $u\ (t)$ is seen to relate to coordinate disturbances, whereas the disturbances that cannot be singled out in this way and affect only the type of transform operators are seen to relate to operator disturbances.

In some situations it is advantageous to evaluate in the plant operator its invariable part, represented by C_z in Fig. 2.2. Commonly this part is taken to be the actuator; then $z\ (t)$ represents its output coordinates which can be treated as the known functions of time.

In handling this problem, the generation of the control function $u\ (t)$ is relegated to the automatic controller, represented in Fig. 2.3 by the operator C_u.

Control theory has devised a number of approaches to the synthesis of C_u. A specific approach and the properties of the control system with a particular controller depend basically on the information available to form the control signal, constraints due to the system operational environment, principal capabilities of the approach concerned, and, finally, performance specifications the control system is to comply.

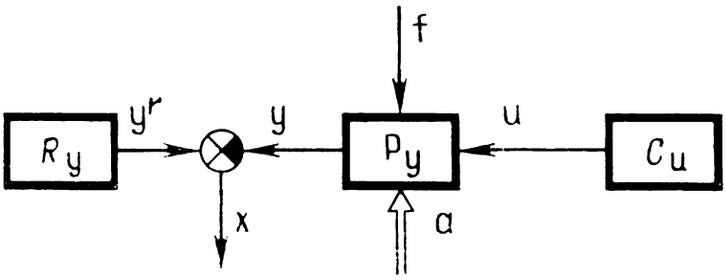


Figure 2.3

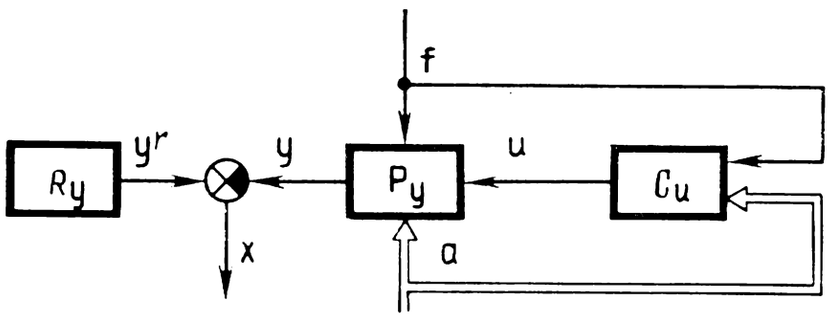


Figure 2.4

Without aiming at a coverage of all techniques in use we take up only some of them. Given a complete information on the plant operator P_y, the function $a\ (t)$ describing its variation in time, values and properties of coordinate disturbances $f\ (t)$, and reference input $y^r\ (t)$, the appropriate form of control function $u\ (t)$ may be achieved on the lines of the *open-loop control principle*, which can be implemented as illustrated in the generalized block diagram of Fig. 2.4 [1-3]. An important feature of this control arrangement is that the function $u\ (t)$ results from the transformations—generally speaking dynamic—of the known functions of time, viz. $a\ (t)$, $f\ (t)$, and $y^r\ (t)$, and is independent of the behavior of the controlled coordinates $y\ (t)$. Speaking another way, the control is given as a program which remains unvariable over the entire period of control. It is quite clear that the absence of at least a part of data necessary for the realization of this control, may render the system inefficient and even inoperable. This limits the area of open-loop control applications.

A greater flexibility is demonstrated by the control based on the information about the current, i.e. actual, status of the controlled system. This type of control is normally formed with the use of the *error-closing control principle* [1, 4, 5], or error-closing control and open-loop control principles used in conjunction [3, 6]. The *closed-loop dynamic systems* or *feedback control systems* that occur in the first case may be represented by the block diagram shown in Fig. 2.5.

The closed-loop systems are seen to differ from open-loop systems primarily in that here the control function directly depends only on the deviation or error. The form of control signal at each instant

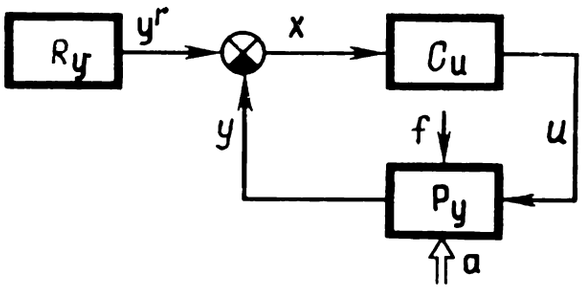


Figure 2.5

is determined automatically as a result of, generally dynamic, transformation of the error signal [26-28]

$$u(t) = u[x(t), t] = C_u(k)\, x(t) \tag{2.2}$$

where C_u is the operator of ordinary or *coordinate feedback*, and k stands for the type of operator C_u. For example, if k represents the parameters of C_u, then $C_u(k)$ implies a parametric family of operators.

In closed-loop control systems, the value of control action at each instant is generated automatically by the feedback operator C_u in accord with the actual behavior of the plant. This salient feature of feedback systems predetermines their fundamental properties.

Feedback systems are well documented at present time. The key problem for these systems concerns the choice of an appropriate feedback operator C_u—as a rule remaining invariant afterwards—which provides the desired properties for the closed-loop system, defined by (2.1) and (2.2).

The fundamental result established in control theory for feedback systems is as follows. Under some rather general assumptions and the absence of constraints on the system coordinates—lifted by a suitable choice of the feedback operator C_u with, generally, variable but bounded parameters—the properties of closed-loop system (Fig. 2.5) may be specified at will. The dependences of the controlled coordinates on the coordinate disturbances $f(t)$, which are not measured directly but belong to a known class of functions, may be driven to any desirable form. Specifically, independence of all, or part of, controlled variables (coordinates) of the coordinate disturbances $f(t)$ can be achieved for all t or for $t \to \infty$ [27-42].

In this connection it would be appropriate to emphasize that for an invariant plant operator P_y, i.e. for $a(t) =$ constant, the operator C_u is also invariant, being uniquely determined by P_y and the class of exterior disturbances. For a variant P_y, i.e. for variable $a(t)$, C_u also becomes time variable. This fact may be represented by assuming the operator-type functions k be time functions of the form $k(t) = k(a(t))$. The immediate implication is that the conversion from open-loop to closed-loop systems enables one to defy

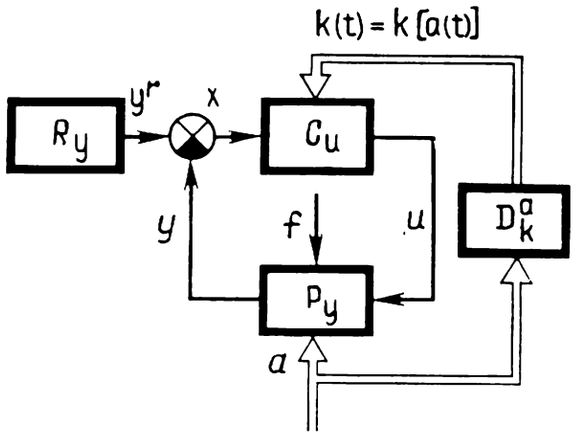


Figure 2.6

the direct incorporation of information on coordinate disturbances $f(t)$, i.e. feedforward for $f(t)$, but to retain the feedforward for operator disturbances $a(t)$, as shown in Fig. 2.6. In other words, the coordinate feedback is an efficient means for dissolving uncertainty in the behavior of the "transparent" plant as caused by the action of unmeasurable coordinate disturbances $f(t)$ belonging to a known class of time functions.

Another level of uncertainty arises when the plant operator varies under the influence of unknown operator disturbances $a(t)$, which can be either variable or constant and take on any value from a certain set. In such circumstances, the feedforward in operator disturbances, shown in Fig. 2.6, cannot be realized. When the operator P_y varies only insignificantly under the influence of operator disturbances, the control problem, as a rule, can be handled to advantage in terms of using coordinate feedback at a fixed C_u. Considerable changes in the plant operator P_y due to operator disturbances $a(t)$ markedly complicate the choice of a suitable feedback in the diagram of Fig. 2.5. Under these conditions, a constant coordinate feedback with a bounded operator prevents a straightforward solution to the control problem, and there is a need for using at various $a(t)$ individual laws of coordinate control, which may differ not only in the form but also in the type of operator C_u involved.

It should be noted that a theoretically exhaustive solution to the problem concerned—for $f(t)$ non-measured and $a(t)$ being arbitrary, bounded, parametric disturbances—is attainable with large amount of feedback [43-45]. The desired result may be achieved in the limit by letting the feedback gain in (2.2) go to infinity. In practical applications, controllers derived in this way, as a rule, cannot ensure closed-loop system stability. With physical constraints on the coordinates of the control system, the closed loop receives oscillations which may also be interpreted as instability of the required control mode.

The main cause of instability in systems with large feedback gain is due to the principal difference between the equations describing the actual controlled plant and those constituting the mathematical model used in choosing the feedback law and computing its parameters. Indeed, the state space dimensionality of an actual process is always larger than that of its model. When the gain parameters in (2.2) are increasing, then starting with some values the aforementioned difference becomes significant and the data used in generating the control signal $u\ (t)$ is no longer sufficient to secure stability. Another familiar cause limiting the practical usage of large gain is due to the physical constraints on the value of control signal, system's state variables, and feedback loop gains.

Concluding this discussion on the properties of closed-loop systems we should recognize that at least in the framework of linear control laws, coordinate feedback applied in large amount is an efficient means of suppressing the negative effects of exterior or coordinate disturbances $f\ (t)$ on system performance. Practical applications of this feedback to compensate for the effects of operator (parametric) disturbances $a\ (t)$ are, however, fairly limited.

2.2 Methods of Control for Ill-Defined Dynamic Systems

The concept of adaptive control has evolved from the constraints on feedback loop gains in ill-defined dynamic systems. The control theory for systems of this class has been extensively developed during a few last decades and now possesses a substantial arsenal of methods handling large class of nonstationary systems operating under uncertainty [12-23, 35, 36, 42, 46-67]. One principal approach widely used in adaptive control consists in estimating, directly or indirectly, the time variable characteristics of the plant related in some manner to plant's parameters and using the derived estimates to change the operator C_u in (2.2) and/or, if possible, also the properties of the plant such that the properties of the closed system are sustained invariable or close to those desired.

Figures 2.7-2.13 represent some block diagrams of adaptive systems, which graphically reflect various methods within the aforementioned approach. The diagrams are borrowed from Ref. 15 and depicted with the conventional symbols adopted in this work. We would like to pinpoint the general traits that will be noted in examining these diagrams. All the adaptive systems concerned either employ a model of the controlled system (Figs. 2.7-2.9) or use frequency response techniques (Figs. 2.10-2.13) with a test-signal generator to estimate the plant gain k_p or its deviation from the specified law. The data derived in this way are employed to correct the con-

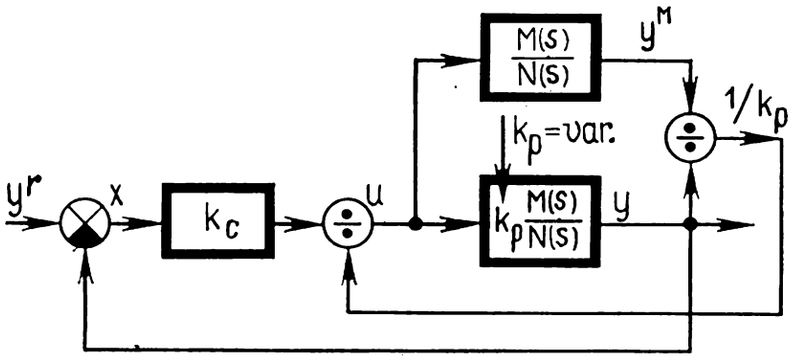
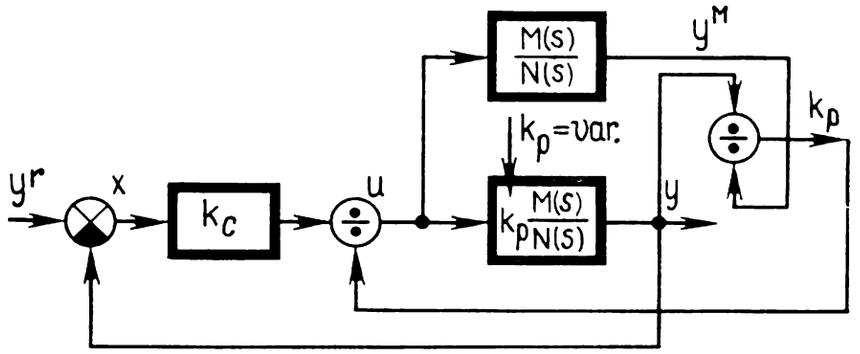


Figure 2.7



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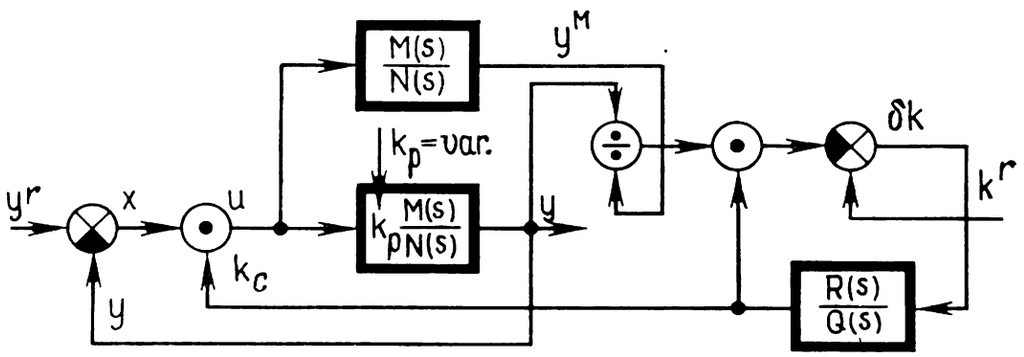

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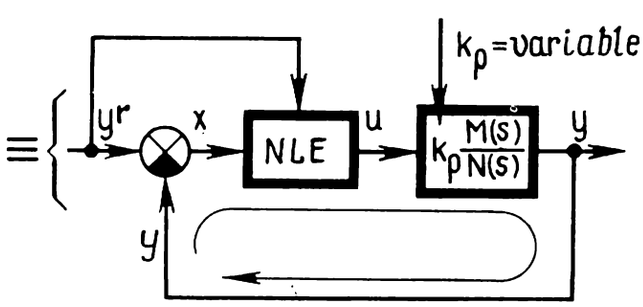

Figure 2.10

troller gain k_c (t). We may, accordingly, conclude that at least the indicated adaptive systems are constructed on two principles of control used in conjunction, namely, error closing control involved in the primary coordinate control loop, and parametric disturbance

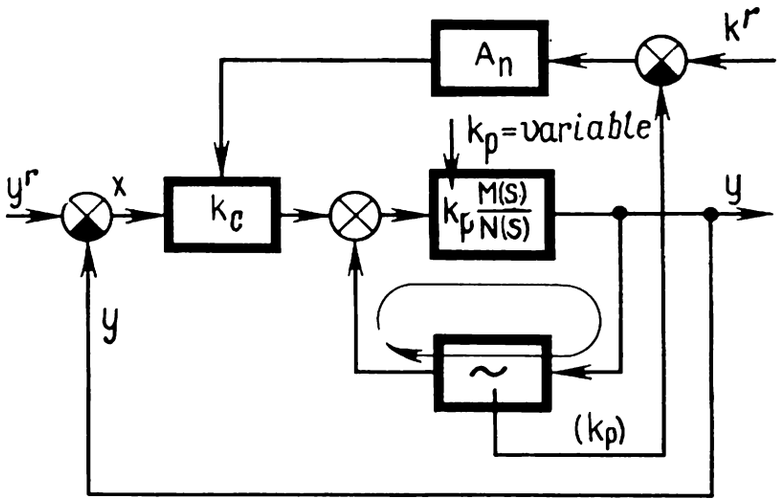

Figure 2.11

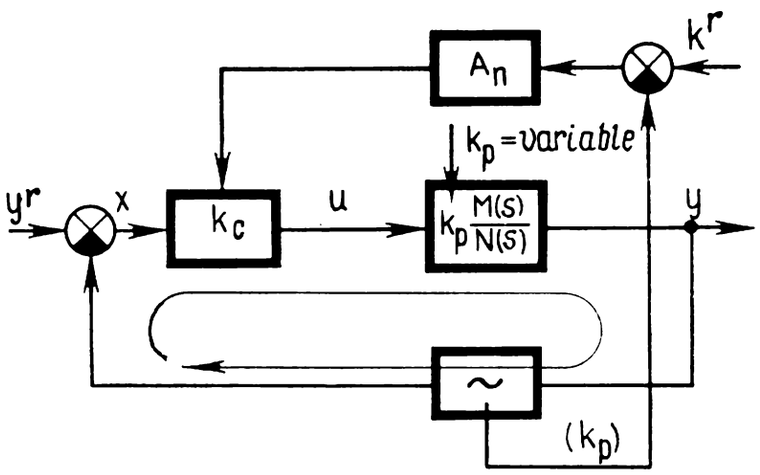

Figure 2.12

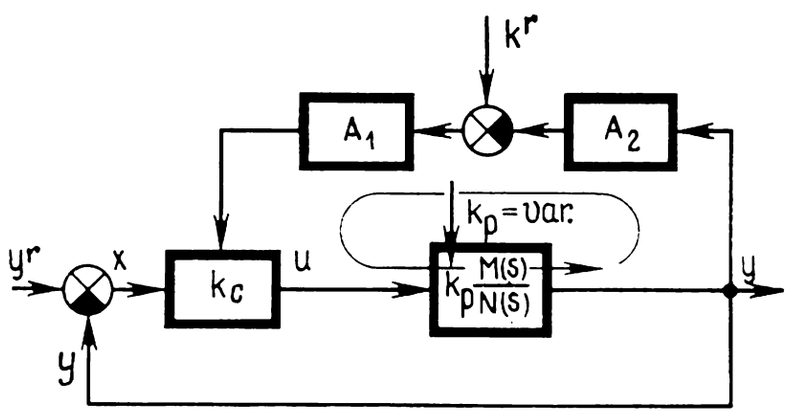

Figure 2.13

control (more specifically disturbance estimate control) involved in handling the parameters of the main loop control law. Such adaptive systems use disturbance control in conjunction with estimation of plant parameters to compensate the effect of parameter variation on the controlled process.

The above conclusion holds not for all adaptive control systems in the same extent as for the block diagrams shown in Figs. 2.7-2.13.

The majority of adaptive systems, however, are characteristic of the dual nature of control actions, as it has been noted, e.g., in Refs. 14, 19. This would justify the application of adaptation techniques to nonstationary system control, in general, only when the plant's characteristics vary "slowly"; this fact is usually formulated as the conditions of quasistationarity [12, 13, 19, 60, 66]. Not in all applications these conditions are readily identifiable, even if they do hold. It should be stressed, however, that these conditions are of much essence not for all adaptive control systems. Frequently, they are merely consequence of the techniques used in adaptation loop synthesis [12, 13].

Now we would like to look at the difference between the adaptive control systems and systems involving high coordinate-feedback gains from a different aspect. Both approaches are usable for synthesis of the control system operating under uncertainty caused, say, by variation of plant's characteristics. Control systems with large gains are designed without any information on plant parameters and external forces, therefore in order to ensure stability as the gains approach infinity the control should receive, generally speaking, an infinite amount of data on the plant.

Adaptive control systems, on the other hand, are built with finite gains in the main loop, therefore control may be effected with the adopted mathematical model and finite amount of data on the system status defined by system's dimensionality. This advantage is gained at the cost of having to obtain some information on the variable parameters of the plant on line. The time taken to derive this information leads to the quasistationary conditions. As a rule, the adaptive approach is preferred when the designer is faced with a high state-space dimensionality.

It will be in order to dwell also on some other methods of handling nonstationary dynamic systems, specifically those based on invariance theory [3, 6, 69-75], on intentional generation of some self-sustaining modes in the control loop [63, 64, 66, 80-82], and on the use of discontinuous controls [24, 25, 83].

The methods of invariance theory either suggest an indirect usage of large gain, or with system parameter varying over large range solve the control problem only in conjunction with an adaptation technique. As a result, their practical application also faces limitations peculiar to adaptive systems and systems with high gain in the main loop.

In self-sustaining mode systems usually implemented within the block diagram shown in Fig. 2.14, the dependence of the processes being controlled on disturbances is diminished by introducing into the closed loop a suitable nonlinear element (NLE) which ensures generation of self-sustaining oscillations. The characteristics of oscillations generated in this way depend on disturbances applied

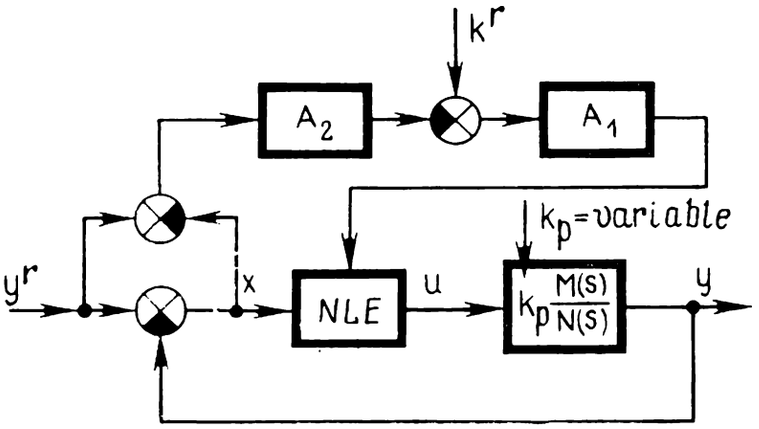

Figure 2.14

to the system and vary along with them. The nonlinearity should be selected such that the characteristics of the oscillations change in the direction where a partial compensation of the disturbances is achieved. The application area for such systems is limited by the presence of intentionally generated oscillations of the processes being controlled, the fact that an asymptotic solution for the control problem is unattainable, a usually insignificant range of compensated parametric variations, and considerable difficulties encountered in analysis of such systems.

In relay (bang-bang) control systems and systems with variable structure operating with plant parameters arbitrarily variable within a limited range and subjected to coordinate disturbances belonging to a sufficiently wide class of functions, the required system behavior and desired response to disturbances can be ensured, as a rule, only when the so-called sliding modes occur in the closed loop. In those situations where the sliding mode is an allowable mode for the primary loop, discontinuous control is an efficient proposition. At the same time, some applications are hard to implement sliding modes in the primary loop, or else the operating conditions of the plant and actuators defy sliding modes altogether.

The consideration given above to the problem of control for ill-defined dynamic systems subjected to uncontrolled operator and coordinate disturbances suggests that it is far from being solved and therefore arises considerable theoretical and practical interest. The research efforts in this field have received a new impetus from the rapidly expanding robotic applications and flexible automated flow lines. As has already been mentioned in the introduction, those solutions are more attractive which take account of the actual constraints imposed on the controlled variables, control action, and gains in various data transmission channels, and allow for continuous control in the primary loop.

3 Generalized Block Diagrams of Binary Control Systems

This chapter will illustrate how the concepts introduced above can be implemented in the structural synthesis of an automatic control system. This system is to handle a nonstationary dynamic process in an uncertain environment which is caused by the action of various types of uncontrolled disturbance, both coordinate and parametric. Each new type of feedback will be directed to overcoming one or another type of uncertainty in controlled system behavior, which is essentially the main purpose of each feedback.

3.1 Constructing Binary Control Systems

The approach on which we shall dwell below has grown largely from the ideas outlined in Refs. 84-88. The problem of control for an ill-defined dynamic system may not be solvable with a fixed linear feedback operator. This exactly will be the focus of our concern in the ensuing text. According to the problem formulation, the feedback operator must be a linear nonstable operator or nonlinear one. No regular methods exist for building such feedback loops. It would then be natural to ask ourselves whether we could forgo the choice of the coordinate feedback (CFB) operator C_u altogether at the system design stage, confining ourselves instead to specifying a certain class of feedback operators and selecting a specific representative within this class with the help of error-closing feedback. By deviation or error we naturally mean a quantity reflecting the deviation of the actual properties from the system performance specifications. In this case the instantaneous value of the feedback operator would be determined by the actual state of the control system depending, of course, on all the disturbances imposed on the system.

Changing from the fixed operator to an operator continuously generated with feedback is natural, moreover, familiar to control theory. Indeed, in feedback systems (Fig. 2.5) such a change is similar to the known transition from the fixed (programmed) control in open-loop systems (Fig. 2.4) to the variable control capable of flexibly responding to sudden changes in the behavior of a controlled variable.

In ordinary systems with coordinate feedback (Fig. 2.5) the need for automatic generation of $u\ (t)$ is due to the lack of information on the coordinate disturbances $f\ (t)$. In the case under consideration this need is in addition due to the lack of information on the plant operator P_y, more specifically, on the operator disturbances $a\ (t)$ which bring about its variation in time. Here we are content to show a methodological difference between the suggested approach and

those employed in the construction of some adaptive control systems, specifically the ones mentioned in the previous chapter.

In the question posed above we probed the possibility of replacing disturbance control, used in adaptive systems to compensate operator (parametric in our case) disturbances, with error-closing control employed for the same purposes. If implementation of this plan manages to avoid estimating current characteristics of the plant, then we may expect that the constraints of the type of quasistationary conditions may be relaxed or eliminated altogether. We could speak of this approach as actually allowing additional capabilities in respect to control of nonstationary dynamic processes under uncertainty. These systems are not to have large gain in the primary loop, and the control function $u\ (t)$ must be a continuous or piecewise continuous function of time. The last requirement often turns out to be quite essential in view of the technicalities of practical actuators. In what follows we shall focus on one method of implementing this idea, which ensures the aforementioned properties for the system.

For convenience of discussion and ease of understanding we shall assume that variation of all operators in time is effected in a parametric way, i.e., that each operator belongs to some parametric family and its specific form is established by specifying its parameters. It would then be natural to refer to the various types of feedback, introduced in Chapter 2, as *coordinate-parametric* (Fig. 1.13*b*), *parametric* (Fig. 1.13*c*), and *parametric-coordinate* (Fig. 1.13*d*).

3.2 Structures of Binary Control Systems with Coordinate-Operator Feedback

It is intuitively clear that many performance specifications for the transient process dynamics in control systems can be expressed as functional relations between system state variables. These relations find their use in control theory in a variety of forms [12, 13, 21, 24, 25, 43, 44, 74, 83]. We shall also assume that the closed-loop system, shown in Fig. 2.5, possesses the necessary dynamic properties also when the plant parameters $a\ (t)$ are variable if at every instant beginning from a certain time the error $x\ (t)$ satisfies the operator equation

$$x\ (t) = R_x\ (c)\ x\ (t) \tag{3.1}$$

where $R_x\ (c)$ is an operator, say differential, whose type and parameters c are selected on the basis of the system performance specifications.

We shall assume that this choice is legitimate and the operator R_x is known. Application of this operator to $x\ (t)$ will obviously yield some function of the type

$$x^r\ (t) = R_x\ (c)\ x\ (t) \tag{3.2}$$

which, generally speaking, is other than $x\ (t)$. We shall call the element with R_x a reference dynamics element and treat the function $x^r\ (t)$ as a reference input formed by this element. The error

$$\sigma\ (t) = \sigma\ (x\ (t)) = x^r\ (t) - x\ (t) \tag{3.3}$$

describes the deviation of the actual behavior of the feedback system (2.1), (2.2) from the specified pattern when $\sigma\ (t) \equiv 0$ and, consequently, (3.1) is met.

In order to compensate this error (3.3) we invoke the principle of error-closing control to establish a negative feedback path with the operator

$$\mu\ (t) = C_\mu\ (\alpha)\ \sigma\ (t) \tag{3.4}$$

The output variable $\mu\ (t)$ will be referred to the operator variables and exploited for changing the parameters of the operator C_u in (2.2) by way of relation

$$k\ (t) = k\ (\mu\ (t)) \tag{3.5}$$

In this case we shall speak of the automatic generation of a coordinate feedback operator C_u with the help of the error-closing control principle.

If we add the coordinate-parametric feedback path (3.4-3.5) into the block diagram of Fig. 2.5, we arrive at the block diagram of Fig. 3.1 which illustrates the use of feedback in the generation of C_u. In this case the CFB system (Fig. 2.5) may be represented as a dynamic element P_x of operator-coordinate type (Fig. 1.10*b*) which plays in the coordinate-parametric feedback loop the role of the controlled plant.

The block diagram in Fig. 3.1 is a typical arrangement for feedback systems. Therefore, choosing the type of operator C_μ and its parameters one may invoke ordinary methods of control theory, say, assume that it realizes two-positional, integral, proportional plus integral, or some other transformation. One should, however, observe the specific character of control actions for the plant P_x, specifically, that these actions are parametric.

It will be noted that it is in principal impossible to provide a constant sign of coordinate-operator feedback (COFB), particularly negative, for the families of linear operators P_y, C_u, C_μ, R_x, and the linear transformations (3.5). The explanation of this fact is that in transforming the error $x\ (t)$ into the control signal $u\ (t)$ with the help of the operator $C_u\ [k\ (\mu)]\ x$ the variable x plays the role of the

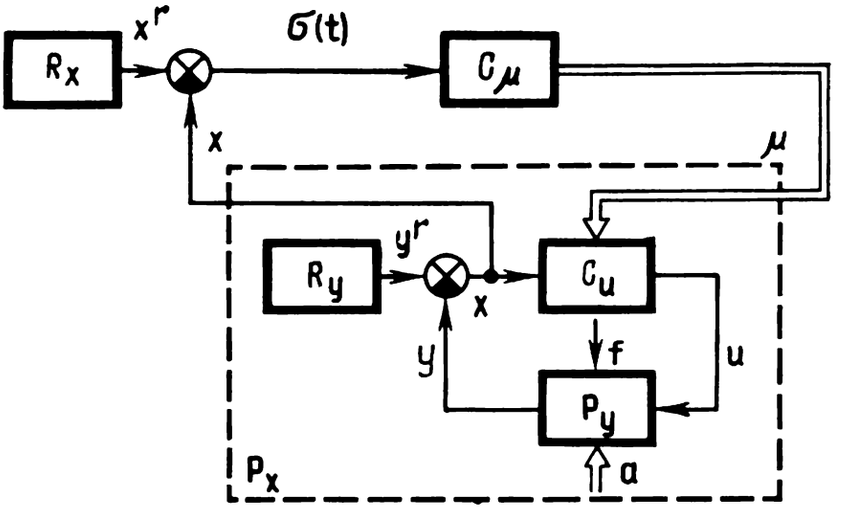

Figure 3.1

gain for the signal μ (t) and as it can be of any sign so will be the COFB. Therefore, at least in the case at hand, we need to ensure the respective change of signs in (3.4) or (3.5) specifying COFB. Saying it another way, the operator C_μ must be nonlinear to have a constant sign for COFB.

Because of the causes outlined in Chapter 1 the parameters in (3.5) must be of bounded variation or else sufficiently small quantities. This is connected with the need to diminish the effects of approximation errors and various constraints on the controlled plant. In what follows we shall assume that these conditions are met, for instance, through a suitable choice of C_μ (α) or the dependence k (μ). Note that no constraints are imposed in advance on the parameter variation rate μ (t). This implies in particular that, in principle, it is possible to organize distinct variation rates for state variables in the CFB and COFB loops. If this rate separation takes place, then these control systems, in contrast to adaptive control systems, may use for control over some time intervals also those fixed values of α in (3.4) at which the closed-loop system (2.1)-(2.2) may go unstable. Later it will be shown that the use of unstable motions over some time intervals is of principal significance.

Let us have a closer look at the difficulties encountered in keeping the desired equation σ $(t) = 0$ for the block diagram shown in Fig. 3.1. Let the operators C_u and C_μ and the function k (μ) be chosen so that beginning from a certain time instant this equation is met and, consequently, the closed-loop control system has the desired properties. The error σ (t) approaching zero implies that the dynamic properties of the closed-loop control system approach the dynamic properties of the reference input R_x to coincide when σ $(t) \equiv 0$. A stepwise variation of plant parameters a (t) changes the properties of the closed-loop control system and subsequent maintaining of the equality σ $(t) = 0$ can be ensured only on the condition that the

parameters k in (2.2) also change stepwise. This is possible only when the σ to μ gains in (3.4) are at infinity, which corresponds to using a large amount of feedback to exactly satisfy the equation $\sigma(t) = 0$.

High gains in the feedback path (3.4) are undesirable as well as in (2.2) because these unavoidably entail the problems characteristic of control systems with infinite gains. Therefore, practical usability of control systems with large amount of COFB is limited in much the same way as that of control systems with large amount of CFB.

With finite gains in (3.4)-(3.5), the parameters $k(t)$ in (2.2) take time to arrive at new values corresponding to new $a(t)$ and may do so after a transient process during which $\sigma(t) \not\equiv 0$. This transient is due to a difference between the dynamic properties of the closed-loop system and those of the reference input R_x caused by the change in the parameters $a(t)$. It is not hard to see that the influence on this transient is more efficient if not only the properties of the closed-loop system are altered by way of COFB but also the respective parameters c of the operator $R_x(c)$ are changed, i.e. operator-type feedback loops are introduced. The objective of control for the parameters of the reference input $R_x(c)$ organized with the help of operator feedback consists in using an additional opportunity for driving closer the dynamic properties of the reference input and the closed-loop control system. These properties when different cause a nonzero error $\sigma(t)$.

3.3 Structures of Binary Control Systems with Coordinate-Operator and Operator Feedback

The discussion of the foregoing section has revealed that keeping the equality $\sigma(t) = 0$ may be facilitated by relating in some way the reference input parameters c with the plant parameters a. By the problem statement on hand, however, the information on the plant parameters $a(t)$ is absent, therefore the required relation can be established only indirectly. We note that in the desired mode, i.e. at $\sigma(t) = 0$, $k(t)$ take the value required to insure the specified performance at arbitrary plant parameters $a(t)$. This implies that the functions $k(t)$ contain information about the functions $a(t)$ and since $k(t)$ and $\mu(t)$ are related by a known dependence, information about $a(t)$ is contained in $\mu(t)$. Therefore, we may expect that instead of nonrealizable direct connection between $c(t)$ and $a(t)$ the required relationship will be achieved by establishing an indirect connection in the form

$$c = c(\rho(t))$$
$$\rho(t) = C_\rho(\beta)\,\mu(t) \qquad (3.6)$$

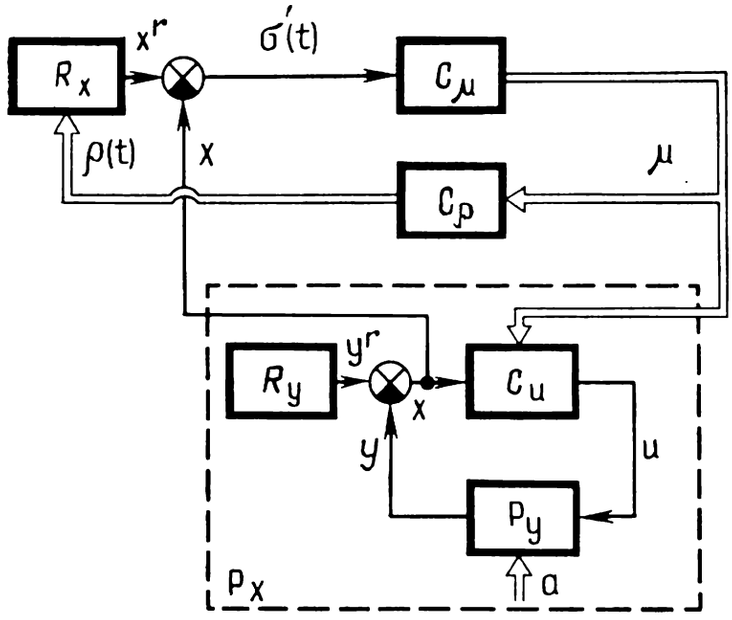

Figure 3.2

Because the variables ρ (t) as well as μ (t) are used to alter the form of operators, in this case R_x (c), it would be natural to relate them to the operator variables. Then C_ρ (β) in (3.6) represents the operator of an operator-type dynamic element (Fig. 1.10*a*), and the relation (3.6) between the variables ρ (t) and μ (t) should be referred to the operator kind, in this case to the parametric relationships. Adding this element in the block diagram of Fig. 3.1 leads to the block diagram (Fig. 3.2) of a nonlinear control system with three types of feedback, viz. CFB, COFB, and OFB.

Inspection of Fig. 3.2 leads us to conclude that OFB, as well as COFB, actually also relies on the error σ (t), but as distinct from COFB this loop has no real process. Consequently, the aforementioned constraints on the choice of C_ρ may be less essential than the constraints on the choice of C_μ. It is important, perhaps, to provide the conditions at which the c parameters of the operator R_x, and hence the dynamic properties of the control system, change only insignificantly when the variables μ (t) vary within the specified limits.

An operator feedback when negative may be thought of as being an additional means of system stabilization at σ $(t) = 0$, when under the aforementioned constraints the coordinate-operator feedback no longer possesses sufficient capabilities for solving this problem. Note that to provide a constant sign for OFB, as well as for COFB, the respective sign reversal in (3.6) must be insured provided that in COFB such change is realized by the dependence (3.5).

3.4 Structures of Binary Control Systems with Coordinate-Operator, Operator, and Operator-Coordinate Feedback

Let us assume that a suitable choice of the OFB operator $C_\rho(\beta)$, its parameters β, and function $c(\rho)$ ensures the equality $\sigma(t) = 0$ beginning from a certain time, and this is maintained under variable plant parameters $a(t)$. This, however, is achieved at the cost of, maybe insignificant, dependence of the dynamic properties of the reference source R_x, and consequently the entire control system, upon the parametric and coordinate disturbances applied to the controlled plant. There may occur a need for eliminating this dependence or shaping it into a desired form. To meet this need, another type of feedback, also built with the error-closing control principle, may be found useful. In what follows we shall show how this feedback can be introduced.

Assume that the dependence of the controlled processes $x(t)$ upon disturbances has a required form provided the parameters $c(t)$ of the reference input R_x satisfy the equation

$$c(t) = R_c(d)\, c(t) \tag{3.7}$$

where R_c is a known operator, and d represents its parameters.

Application of R_c to the actual function $c(t)$ yields a function $c^r(t)$ which, in general, is distinct from $c(t)$.

Assume R_c being a reference input element and $c^r(t)$ the reference action to form the error

$$\nu(t) = c^r(t) - c(t) \tag{3.8}$$

describing the deviation of the actual variation of R_x parameters from the behavior specified by (3.7). Construct a control loop to compensate for the error (3.8). Introduce an operator C_v transforming the parametric error $\nu(t)$ into the variable

$$v(t) = C_v\, \nu(t) \tag{3.9}$$

which will be added to the main control signal $u(t)$. In the nomenclature introduced earlier in this text, such an operator takes the image of an operator-coordinate element (Fig. 1.10*b*), in this case specifically, of a parameter-coordinate element.

Augmenting the block diagram in Fig. 3.2 by the path (3.9) transfers it to the block diagram shown in Fig. 3.3. Let the entire control system depicted in Fig. 3.2 be thought of as a certain dynamic element P_c of coordinate-operator type, then Fig. 3.3 provides a vivid representation of the use of operator-coordinate feedback (Fig. 1.13*c*) in the synthesis of the specified control system. That is why (3.9) is referred to as *operator-coordinate feedback*. The object of OCFB

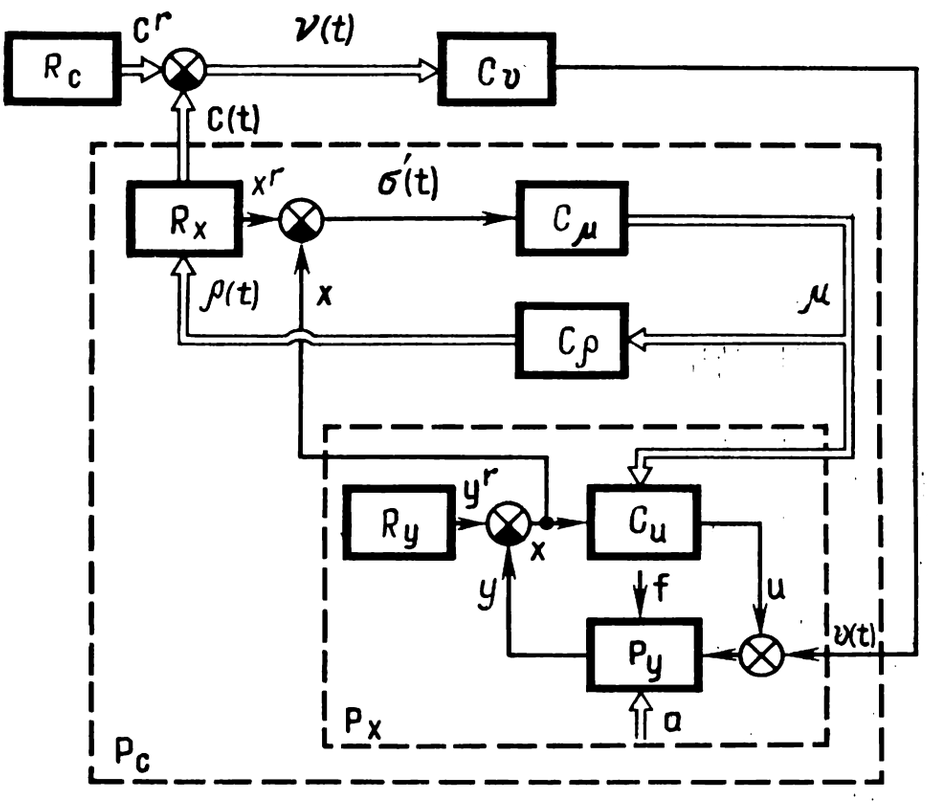

Figure 3.3

is to insure $\nu\ (t) \to 0$ as $t \to \infty$. The effect of this feedback may be attributed to an additional nearing of the dynamic properties of the reference input R_x and the closed-loop control system as achieved with the action of $v\ (t)$ on the controlled plant.

The block diagram shown in Fig. 3.3 represents a control system incorporating all types of feedback introduced in Chapter 2. Our aim now will be to demonstrate how the use of different types of familiar feedback operator leads to new nonlinear control algorithms providing the closed-loop control system with the properties and characteristics that have been discussed above.

Concluding the chapter we are bound to note that some types of feedback have already been reported in the literature [12, 13, 15-18, 77, 89, 90] and employed to alter transform operators. They may be used to advantage in describing, designing, and analyzing nonlinear dynamic systems.

II Free Motion Control

4 Definitions and Nomenclature

This part covers the methods of control—both familiar and derived with the approach outlined in the previous part—for nonstationary dynamic processes described by one differential equation of the form

$$\ddot{y}(t) + a_2(t)\,\dot{y}(t) + a_1(t)\,y(t) = u(t), \quad t \geqslant t_0 \tag{4.1}$$

where t is the current time and t_0 is the initial time instant (time zero).

All functions in (4.1) are scalar functions of time, defined for all $t \geqslant t_0$. Functions $a_1(t)$ and $a_2(t)$ will be called process *parameters*. It will be assumed that $a_i(t)$, $i = 1, 2$, are arbitrary piecewise continuous and unknown functions defined for all $t \geqslant t_0$. It is assumed that only bounds are known for these functions a_i, viz.

$$a_i^- \leqslant a_i(t) \leqslant a_i^+, \quad i = 1, 2; \quad t \geqslant t_0 \tag{4.2}$$

where $a_i^{\pm}$, $i = 1, 2$ are given values, positive or negative.

As can be seen from (4.1) the situation on hand involves no coordinate disturbances $f(t)$, i.e., $f(t) \equiv 0$ for $t \geqslant t_0$. It will be also assumed in the chapters of this part that $y^r(t) \equiv 0$ for $t \geqslant t_0$ (see Fig. 2.1). These conditions define the control problem of *free motion* or *free oscillation* of a dynamic process (4.1)-(4.2) subjected to the action of unmeasurable parametric disturbances $a_1(t)$ and $a_2(t)$. The content of the control problem for the process (4.1)-(4.2) will be given a separate examination.

Any actual dynamic process (actual dynamic system) which changes its behavior under the influence of control actions $u(t)$, $t \geqslant t_0$, will be called a *controlled system* or *plant* and denoted in block diagrams as P_y.

A scalar or vector variable $y(t)$ will be called a *controlled variable*, or *controlled coordinate*.

When the model of a controlled plant is written as one differential equation in $x(t)$ describing the deviation $x(t) = y^r(t) - y(t)$ between the scalar functions $y^r(t)$ and $y(t)$, then this model will be called an error-closing controlled process and designated by $\mathcal{P}_x$—

not to be confused with the symbol P_x used above (Figs. 3.1-3.3) to represent dynamic elements of operator-coordinate type with input $\mu(t)$ and output $x(t)$. The scalar function $x(t)$ is called the *error signal* or *error coordinate*. In this nomenclature, for (4.1) the $\mathcal{P}_x$-process equation is as follows:

$$\ddot{x}(t) + a_2(t)\dot{x}(t) + a_1(t)x(t) = -u(t), \quad t \geqslant t_0$$

If the model of a controlled plant or system is given by a system of first order differential equations in state variables, then this model will be called the S-system. For example, in the state variable terminology of $x_1(t) = x(t)$ and $x_2(t) = \mathrm{d}x(t)/\mathrm{d}t = \dot{x}(t)$ equation (4.1) may be written as an S-system for $t \geqslant t_0$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -a_1(t)x_1(t) - a_2(t)x_2(t) - u(t)$$

It will be noted that the $\mathcal{P}_x$-processes are a particular case of the S-systems. Therefore, in what follows the definitions and concepts will be given for the S-systems and the stepdown modification for the $\mathcal{P}_x$-processes is as a rule self evident.

The vector function $\boldsymbol{x}(t) = [x_1(t), x_2(t)]^{\mathrm{T}} \in \mathbb{R}_x^2$ will be referred to as a *controlled process*. Here and below the superscript T stands for the transpose.

Below we shall assume that every control function $u(t)$ not given in explicit form for $t \geqslant t_0$ belongs to the class of functions $\mathcal{U} = \mathcal{U}(t_0, \infty)$ for which giving an initial vector $\boldsymbol{x}(t_0)$ and a function $u(t) \in \mathcal{U}$ defines, for $t \geqslant t_0$, an S-system solution $\boldsymbol{x}(t) = \boldsymbol{x}[t; t_0, \boldsymbol{x}(t_0)]$ continued over the entire semiaxis $t \geqslant t_0$ and such that $\boldsymbol{x}[t_0; t_0, \boldsymbol{x}(t_0)] = \boldsymbol{x}(t_0)$.

In constructing a control system it is normally assumed that the system designer has some a priori information on the system to be controlled. We shall denote this information I. For an S-system, for instance, the available prior information may be contained in the set

$$I = \{\boldsymbol{x}(t), \quad t \geqslant t_0; \quad a_i^{\pm}, \quad i = 1, 2\}$$

The relationships defining the transformation of I into a control $u(t)$ will be called a *control algorithm*, denoted by $\mathcal{A}$ and referred to as the $\mathcal{A}$-algorithm. This algorithm is to be realized in a physical device, a *controller*. In block diagrams the $\mathcal{A}$-algorithm is represented by the symbol of the respective transformation operator, e.g., C_u in Figs. 2.1-2.5.

The controlled plant with the associated controller (controlling system), modeled as an S-system with a control algorithm, constitutes the *control system*.

If the manipulated variable $u(t)$ is a "program", i.e. a time function predetermined for all $t \geqslant t_0$, then the control system is called the *open-loop* system and will be denoted by S_o (see Fig. 2.4).

If for each $t \geqslant t_0$ the control $u(t)$ is formed by feedback, say as a function $u[t, x(t)]$, then the system is called the *closed-loop* system, denoted by S_c, and referred to as the S_c-system (see Fig. 2.5).

The choice of a control algorithm by the S-system will be called the synthesis of a control system.

We shall say that the control system solves the *control problem* (the regulation problem when $y^r(t) \equiv 0$ for $t \geqslant t_0$) asymptotically if for each controlled process $x(t) = x[t; t_0, x(t_0)]$ $\lim_{t \to \infty} \| x(t) \| = 0$, where double bars stand for a norm.

The control system solves the control problem Δ-accurately (for some constant $\Delta > 0$) if for every controlled process $x(t)$ there exists a time instant $t_* = t[x(t_0), \Delta] \geqslant t_0$ such that $\| x(t) \| \leqslant \Delta$ for all $t \geqslant t_*$.

Every $\mathcal{A}$-algorithm specifies, generally speaking, a parametric class of control algorithms and forms a control function $u(t)$, belonging to a class of function $\mathcal{U}$, using an information I on the S-system. The possibility of solving the control problem on hand in either of above-defined senses by means of a specific $\mathcal{A}$-algorithm will be defined as follows.

The S-system will be called *$\mathcal{A}$-controlled* over a class of control functions $\mathcal{U}$ subject to the available information I if $u(t) \in \mathcal{U}$ and the respective control system solves the control problem asymptotically.

The fact that the aforestated conditions are satisfied for an S-system will be symbolically written as

$$S \in \mathcal{C}_{\mathcal{A}}\{\mathcal{U}, I\} \tag{4.3a}$$

It is far from always that the control problem is solvable asymptotically. Situations often occur when the controlled processes go stable in the vicinity of a found solution. The size of this neighborhood determines the error of the solution. Situation like this will be characterized in terms of the definition given below.

The S-system will be called $(\Delta, \mathcal{A})$-controlled over a class of control functions $\mathcal{U}$ under the available information I if $u(t) \in \mathcal{U}$ and the respective control system solves the control problem Δ-accurately. In symbolic form this statement will be written as

$$S \in \mathcal{C}_{(\Delta, \mathcal{A})}(\mathcal{U}, I) \tag{4.3b}$$

The quantity Δ appearing in (4.3b) may turn out to depend on $\mathcal{A}$-algorithm parameters and can be chosen at will including as small values as desired. This possibility is outlined in the definition that follows.

The S-system will be called $\mathcal{A}$-quasicontrolled if for any $\Delta > 0$ there exist $\mathcal{A}$-algorithm parameters such that render the S-system $(\Delta, \mathcal{A})$-controlled. A shorthand notation for this fact is as follows

$$S \in \mathcal{QC}_{\mathcal{A}}(\mathcal{U}, I) \tag{4.3c}$$

where the letter $\mathcal{Q}$ stands to distinct it from (4.3a).

The above definitions tell of the controlled processes converging to zero or to some, specified in case of (4.3c), neighborhood of zero. More often than not a stronger requirement is imposed, commonly focused on stability for control systems. Therefore, if in addition to (4.3a) the controlled processes $x(t)$ exhibit the property that for any initial value $x(t_0)$

$$\| x[t;\ t_0,\ x(t_0)] \| \leqslant N \| x(t_0) \| \tag{4.4a}$$

where N is a positive constant independent of $x(t_0)$ and t_0, then the S-system will be called *steady $\mathcal{A}$-controlled*. This is stated in shorthand as

$$S \in \mathcal{C}_{\mathcal{A}}(\mathcal{U}, I;\ N) \tag{4.5a}$$

For Δ-accurate solution of the control problem, the estimate (4.4a) may not be satisfied. In order to establish the property similar to (4.5a) we shall exploit the following procedure.

Let $\overline{U}_\Delta$ denote a sphere of radius Δ centered on the origin O_x, i.e.,

$$\overline{U}_\Delta = \{x \in \mathbb{R}^n_x : \| x \| \leqslant \Delta\}$$

The distance $r(x, \overline{U}_\Delta)$ of an arbitrary point $x \in \mathbb{R}^n_x$ to the sphere $\overline{U}_\Delta$ will then be defined as

$$r(x, \overline{U}_\Delta) = \min_{x' \in \overline{U}_\Delta} \| x - x' \|$$

We will say that the S-system is *steady* $(\Delta, \mathcal{A})$*-controlled* (*steady $\mathcal{A}$-quasicontrolled*) if in addition to (4.3b) (or, respectively, to (4.3c)) for any controlled process $x[t;\ t_0,\ x(t_0)]$ and a certain constant N independent of t_0

$$\| x[t;\ t_0,\ x(t_0)] \| \leqslant N[\Delta + r(x(t_0), \overline{U}_\Delta)],\quad t \geqslant t_0 \tag{4.4b}$$

In symbolic form this may be written for the first case as

$$S \in \mathcal{C}_{(\Delta, \mathcal{A})}(\mathcal{U}, I;\ N) \tag{4.5b}$$

and for the second case as

$$S \in \mathcal{QC}_{\mathcal{A}}(\mathcal{U}, I;\ N) \tag{4.5c}$$

It is not hard to see that at $\Delta = 0$ inequality (4.4b) coincides with (4.4a) under the above definitions of distance $r(x, \bar{U}_\Delta)$.

When the controlled processes satisfy (4.4) for any $t \geqslant t_0$ and

$$\| x [t, t_0, x(t_0)] \| \leqslant N \| x(t_0) \| \exp [-\delta (t - t_1)] \tag{4.6}$$

for any $t \geqslant t_1$, where $t_1 = t_1 [t_0, x(t_0)]$ is some moment of time, and N and δ are positive constants independent of $\| x(t_0) \|$ and t_0, then we say that the S-system is *exponentially $\mathcal{A}$-controlled*, and put down this fact in shorthand as

$$S \in \mathcal{C}_{\mathcal{A}}(\mathcal{U}, I; N, \delta) \tag{4.7}$$

If, finally, the aforestated properties hold true not for any $x(t_0) \in \mathbb{R}^n$ but rather only for $x(t_0)$ of some domain B we shall indicate this fact by adding the subscript B to the respective shorthand expression, say in place of (4.7) in this case we write

$$S \in \mathcal{C}_{\mathcal{A}}(\mathcal{U}, I; N, \delta)_B \tag{4.8}$$

In the subsequent discussion we shall evaluate various approaches to the synthesis of control algorithm which supply the closed-loop control system with the properties expressed by equations (4.3)-(4.8). In building and comparing different closed-loop systems we take account of the constraints imposed on the magnitudes of control signal and gains in various control loops. We also tend to diminish the dependence of controlled process properties upon the S-system ($\mathcal{P}_x$-process) parameters $a_1(t)$ and $a_2(t)$ for $t \geqslant t_0$.

We note in conclusion that in this chapter the information on the controlled plant is contained in one of the sets stated either as

$$I_2 = \{x(t), \dot{x}(t), t \geqslant t_0; \quad a_i^{\pm}, i = 1, 2\}$$

or as

$$I_1 = \{x(t), t \geqslant t_0; a_i^{\pm}, i = 1, 2\}$$

All we have said above defines the general formulation of the control problem discussed in this text. A particular problem formulation differs from the general one by accounting for various types of constraint, imperfections in system element implementation, error in the information acquired, etc. Each time the features accounted in the problem solving will be defined in detail.

5 Time-Invariant Linear Coordinate Feedback

Let us look at the potentialities the time invariant linear coordinate feedback (CFB) offers in solving the problem of $\mathcal{P}_x$-process control and discuss the peculiarities arising in such a closed-loop control system owing to the constraints on the magnitude of control signal and CFB loop gain. The major objectives of control we set concern themselves with attainment of the exponential stability for the origin of the $\mathbb{R}_x^2$ phase space, O_x (i.e. in terms of the above definitions, to get the system exponentially $\mathcal{A}$-controlled at certain $\mathcal{A}$) and with diminishing the dependence of controlled process behavior on the $\mathcal{P}_x$-process parameters, or $\mathcal{P}_x$ parameters for short.

It is not hard to see that for the $\mathcal{P}_x$-process, time-invariant proportional feedback with a control algorithm of the type $u[x(t)] = kx(t)$, where k is a constant, is incapable of achieving the control objectives we have set above.

Far greater capabilities are offered by the familiar proportional plus derivative control algorithm given as

$$\begin{aligned} u[x(t)] &= k\sigma(t) \\ \sigma(t) &= c\dot{x}(t) + x(t), \quad t \geqslant t_0 \end{aligned} \tag{5.1}$$

where k and c are some positive constants. We shall refer to this algorithm symbolically as $\mathcal{A}_k$. The block diagram associated with

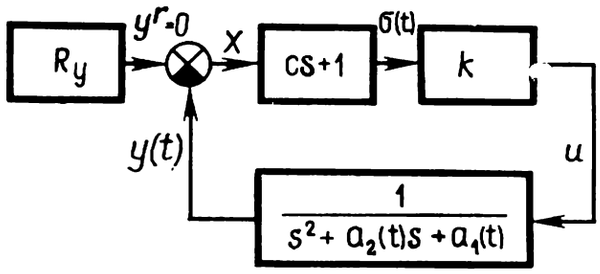

Figure 5.1

the respective closed-loop control system is given in Fig. 5.1, while the equation describing the system behavior has the form

$$\ddot{x}(t) + [a_2(t) + kc]\dot{x}(t) + [a_1(t) + k]x(t) = 0, \quad t \geqslant t_0 \tag{5.2}$$

5.1 Finite Gain

In this section we evaluate the properties of control systems with $\mathcal{A}_k$-algorithms on the assumption that the gain k in (5.1) is bounded from above by a certain value k_0.

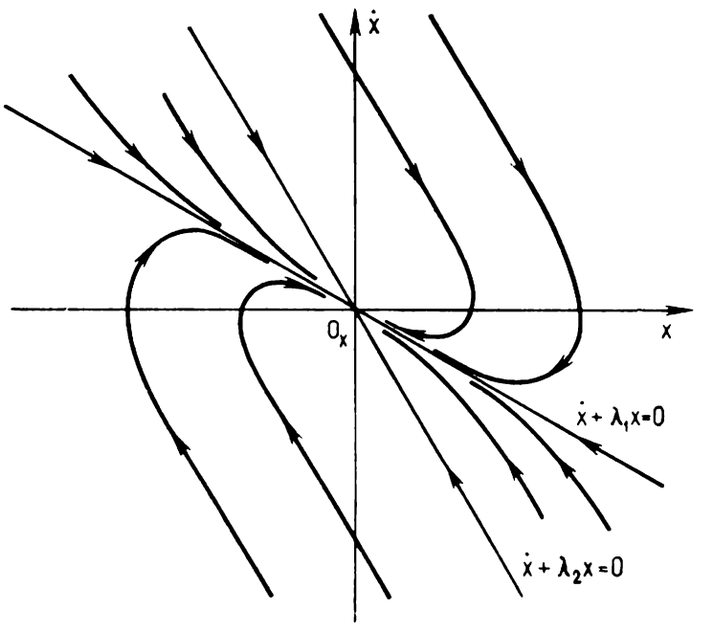

Figure 5.2

It is an easy matter to see that for any pair of a_1 and a_2 fixed within their variation range (4.2), given the inequality

$$(a_2 + kc)^2 > 4\,(a_1 + k) \tag{5.3}$$

is satisfied, the qualitative phase portrait for the bounded-k closed-loop control system is as shown in Fig. 5.2. To distinct pairs (a_1, a_2) there correspond, generally speaking, different positions of the asymptotes $\dot{x} + \lambda_1 x = 0$, $\dot{x} + \lambda_2 x = 0$ and different phase trajectories of the system (5.2). Consequently, the properties of controlled processes in such a system depend on the $\mathcal{P}_x$ parameters. The controlled processes $x\,(t) = [x\,(t), \dot{x}\,(t)]$ may exhibit oscillatory behavior if the upper bound on the gain k is such that the condition (5.3) is violated for certain a_1 and a_2. In addition, for arbitrary variation of $a_1\,(t)$ and $a_2\,(t)$ within (4.2) the very subject of stability for the origin O_x [note that the point $(x = 0, \dot{x} = 0)$ is the equilibrium position of the system (5.2)] needs a separate investigation. The wider the variation ranges of $\mathcal{P}_x$ parameters, the more pronounced at a given k_0 appear the aforementioned setbacks. It may, therefore, be concluded that the linear time-invariant feedback of bounded gain fails to solve the problem of $\mathcal{P}_x$-process control in the aforementioned sense.

5.2 High Feedback Gain

Assume now that the gain k in (5.1) is no longer constrained and look at the possibilities the $\mathcal{A}_k$-algorithm offers in solving the control problem under these conditions. It is well known

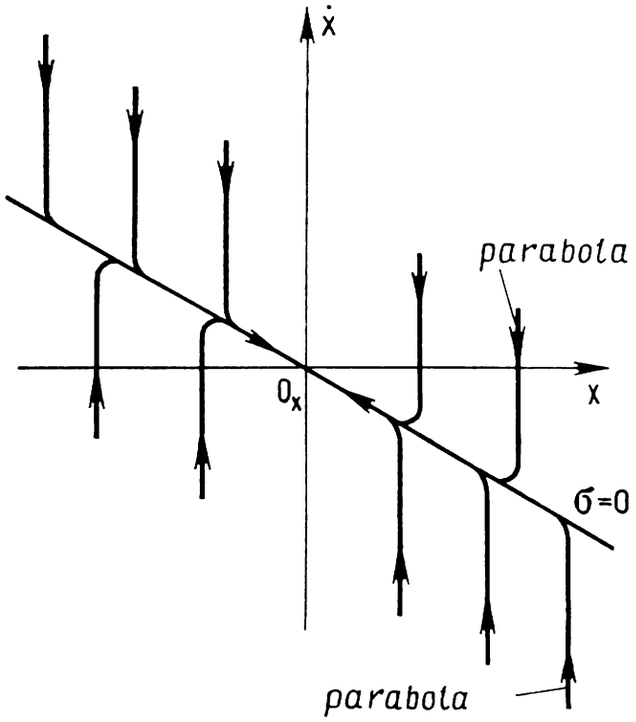

Figure 5.3

that the stability for the point O_x and complete independence of the controlled processes of the parameters $a_1(t)$ and $a_2(t)$ can be achieved with control algorithm (5.1) by way of increasing k to infinity [44, 91], that is by introducing large amount of negative feedback. With k at infinity, the phase portrait of the closed-loop system (5.2) looks for any $a_1(t)$ and $a_2(t)$ of (4.2) as indicated in Fig. 5.3.

In such a control system, any controlled process with the initial conditions $[x(t_0), \dot{x}(t_0)]$ almost in no time appears on the line $\sigma(x) = c\dot{x} + x = 0$, and from this moment onwards its first component varies as $x(t) = x(t_0) \exp[-(t - t_0)/c]$. A mathematically rigorous proof of this fact can be given with the ideas and methods devised by Tikhonov [79, 89]. Theoretically at least, this statement implies that a high-gain feedback of the type (5.1) is capable of solving the control problem at hand. Denoting the considered control algorithm by $\mathcal{A}_\infty$ we arrive, in terms of the notation introduced earlier, at the following shorthand writing for this statement

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\infty}\{\mathcal{F}_c,\ I_2;\ (1+1/c),\ 1/c\}$$

where $\mathcal{F}_c$ is the set of continuous functions of time. In words, this statement implies that the $\mathcal{P}_x$-process on hand is exponentially $\mathcal{A}_\infty$-controlled and all the controlled processes are independent of $\mathcal{P}_x$ parameters whatever large their variation range may be, provided

it is bounded. However, practical implementation of this method is denied by a number of causes. The effect of some of these causes was discussed qualitatively in Part One of the book. Later these topics will be elucidated in more detail.

5.3 Constrained Control Signal

In actual control systems there is always a constraint imposed on the magnitude of the control signal $u(t)$. Let us evaluate how profoundly such a constraint could affect the statement of the preceding section. Assume that for a positive constant M this constraint is given as

$$\begin{aligned} u &= M \quad \text{for} \quad k\sigma \geqslant M \\ &= k\sigma \quad \text{for} \quad |k\sigma| < M \\ &= M \quad \text{for} \quad k\sigma \leqslant -M \end{aligned} \tag{5.4}$$

It is quite obvious that at $k = \infty$ an amplifier of gain k cascaded with a limiter (5.4) is equivalent to a two-position element (Fig. 5.4) whose action is described by

$$u = M \operatorname{sgn} \sigma \tag{5.5}$$

where sgn $(\cdot)$ is the sign function.

Thus, the limitation (5.4) in the control path makes the control system (Fig. 5.1) with large amount of negative feedback equivalent to the control system with on-off feedback depicted in Fig. 5.5. A feasible phase portrait for such a closed-loop control system drawn qualitatively for some "frozen" $\mathcal{P}_x$ parameters is presented in Fig. 5.6. The differential equation describing this system behavior for $t \geqslant t_0$ is as follows:

$$\ddot{x}(t) + a_2(t)\,\dot{x}(t) + a_1(t)\,x(t) = -M \operatorname{sgn} \sigma\,(x(t)) \tag{5.6}$$

It relates to the class of differential equations with a discontinuous right-hand side [92]. To define the solution to this type of differential equation, specifically equation (5.6), we shall exploit the definition given in Ref. 93. The results of this publication will also be invoked to evaluate the existence of such solutions and their continuation on the entire semiaxis $t \geqslant t_0$.

The controlled process behavior in such a closed-loop system is known [92] to exhibit oscillatory character, depends on $\mathcal{P}_x$ parameters, and on the line $\sigma(x) = 0$ in the vicinity of the quiescent point O_x there may exist a sliding-mode segment 1 to 1′. In agreement with Ref. 93, the differential equation describing the control system behavior in the sliding mode has the form

$$\dot{x}(t) = -(1/c)\,x(t) \tag{5.7}$$

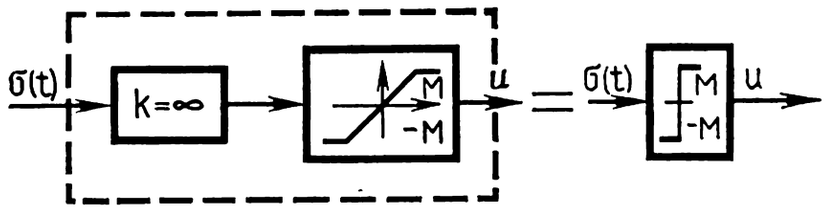

Figure 5.4

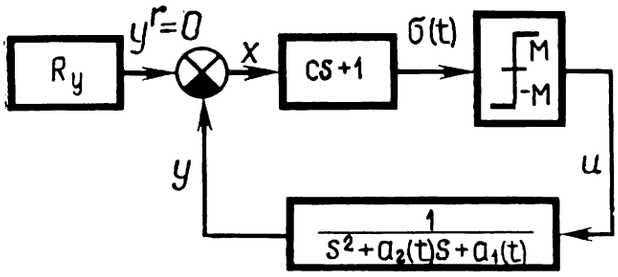

Figure 5.5

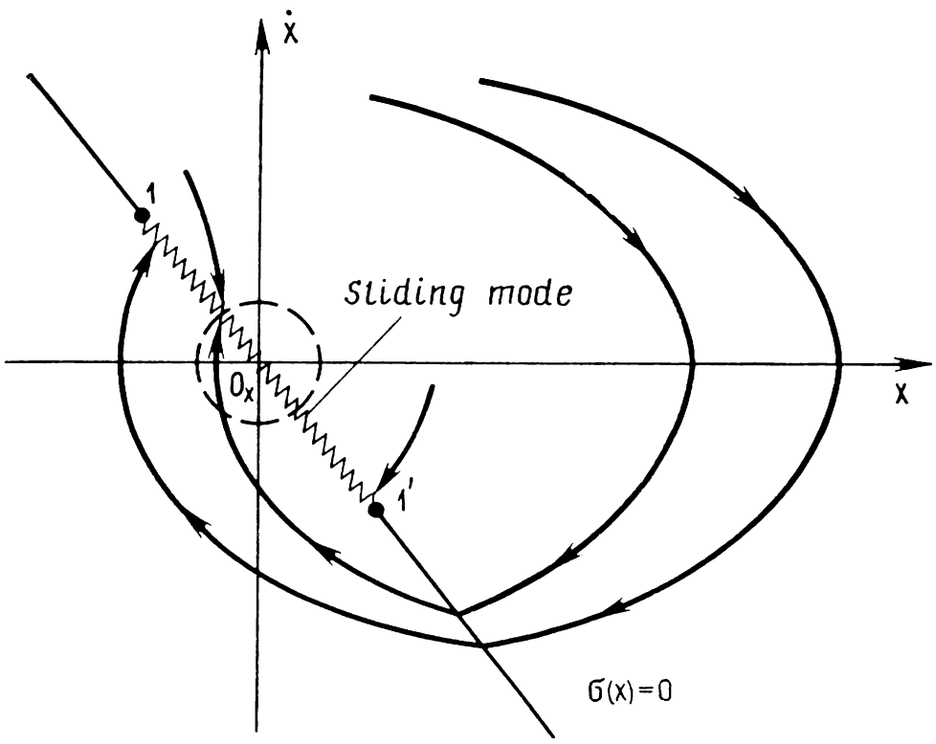

Figure 5.6

Accordingly, as soon as a sliding mode occurs at time t' on the line $\sigma(x) = 0$ the controlled processes of the system on hand will be governed by equation (5.7) and for $t \geqslant t'$ their first component will obey $x(t) = x(t') \exp[-(1/c)(t - t')]$. Hence, starting with this time instant the controlled processes will be independent of the $\mathcal{P}_x$ parameters $a_1(t)$ and $a_2(t)$.

A segment of sliding mode, if existent, is always finite, its length depending on $\mathcal{P}_x$ parameters and the magnitude of M. The "stronger" the limitation on control, the shorter the line section, and the lesser

portion of the transient process is independent of $\mathcal{F}_x$ parameter variation.

Denoting by the letter $\mathcal{M}$ the set of points of the line $\sigma(x) = 0$ common for all the sections [1, 1'] of the sliding mode at different pairs (a_1, a_2) within (4.2), i.e.,

$$\mathcal{M} = \left\{(x, \dot{x}): |x| \leqslant \frac{Mc^2}{\sup\limits_{t \geqslant t_0} |1 - ca_2(t) + ca_1(t)|}, \; c\dot{x} + x = 0\right\}$$

by $\mathcal{F}_s$ the family of time functions discontinuous in each point, and by $\mathcal{A}_{\text{b-b}}$ the bang-bang control algorithm (5.5) allows us to summarize the aforelisted properties of on-off control systems in shorthand as follows:

$$\mathcal{F}_x \in \mathcal{C}_{\mathcal{A}_{\text{b-b}}}\{\mathcal{F}_s, I_2; (1 + 1/c), 1/c\}_{\mathcal{M}}$$

Thus, with some constraints imposed on the magnitude of control signal, high negative feedback gain cannot provide the desirable behavior of controlled processes in the whole space (Fig. 5.3). The independence of parameters and the exponential stability with the desired exponent are achieved for the controlled processes beginning only with the moment when a sliding mode occurs, i.e., subject to the initial conditions from the set $\mathcal{M}$ (Fig. 5.6).

Other causes constraining the actual possibilities of high feedback gain in solving the control problem at hand include a variety of imperfections in the realization of control laws, inaccuracies incurred in the mathematical description of the plant to be controlled, and the like. Below we shall dwell on some of them in more detail.

5.4 Imperfections

This section will be devoted to the effect of a certain type of imperfections on the performance of a system controlling a $\mathcal{F}_x$-process by means of an $\mathcal{A}_k$-algorithm with high gain in the coordinate feedback loop and some limitations imposed on the control signal. In our evaluation we adopt a qualitative approach relying on on-off control system developments.

In the previous section we have established that given a constraint on the magnitude of control signal, when the $\mathcal{A}_k$-algorithm gain grows to infinity, i.e. $k = \infty$, the controlled process behavior coincides with that of the on-off control system controlled by the algorithm (5.5). It is well known that in practice such two-position switchings cannot be realized perfectly. In some applications the effect of imperfection in a realization of the on-off action is negligibly small from the practical aspect and may be safely neglected, whereas

in others it can be rather significant and introduces a qualitative change into the properties of the control system. Whatever small the effect of such imperfection may be from the practical viewpoint, it would be rather interesting to evaluate the principal features appearing under these circumstances in the control system behavior.

For definiteness we assume that the imperfection in bang-bang switching is associated with a dead time characterized by a constant $\tau > 0$. In other words, a switching of a two-position element occurs not at the moment its input coordinate $\sigma(t)$ becomes zero, but rather after a delay τ. The control system with such an element is described by difference-differential equations, and its phase space is an infinite-dimensional functional space. An exhaustive analysis of all controlled process properties in such a system is a fairly complicated task, therefore one normally confines himself to looking at some particular, though typical, classes of controlled processes [92, 94, 95].

The more natural and traditional assumption is that zeros of $\sigma(t)$ are spaced on the time axis at least as far as the dead time τ. This assumption singles out a class of controlled processes whose behavior is of greatest significance from the practical standpoint. For the on-off control system on hand, this assumption allows using the phase space $(x, \dot{x})$ to represent processes of this class [92].

For control systems similar to those evaluated in this section, in-depth analyses performed under the above assumption are reported in Refs. 92, 94, 95. It has been established that for sufficiently small τ, the closed-loop control system outlined in Section 5.3 retains the general character of controlled process behavior, but there occur a number of essential deviations. In the control system with dead-time switching, an ideal sliding mode gives way to the so-called real sliding mode with a finite deviation of the controlled processes from the switching line $\sigma(\boldsymbol{x}) = 0$ and a finite switching frequency for the two-position element. In addition, in the vicinity of the origin, nondecaying oscillations of the controlled processes take place, and with fixed $\mathcal{P}_x$ parameters a limit cycle can occur in the vicinity of O_x [92, 94, 95].

Figure 5.7 shows a typical phase portrait of such control system behavior. More specifically, it presents to an increased scale the neighborhood of O_x indicated by a dashed circle in Fig. 5.6. In Fig. 5.7, $\sigma'(\boldsymbol{x}) = 0$ and $\sigma''(\boldsymbol{x}) = 0$ are the switching lines. We note that the parameters of a limit cycle, when it occurs, the position of the lines $\sigma'(\boldsymbol{x}) = 0$ and $\sigma''(\boldsymbol{x}) = 0$, and the size of the area where controlled process behavior is similar to that illustrated in Fig. 5.7, all of them depend on, and are governed by, the $\mathcal{P}_x$ parameters, and the dead time.

Similar conclusions may be drawn for other types of imperfections, as occurring in the realization of on-off function, or for dynamic

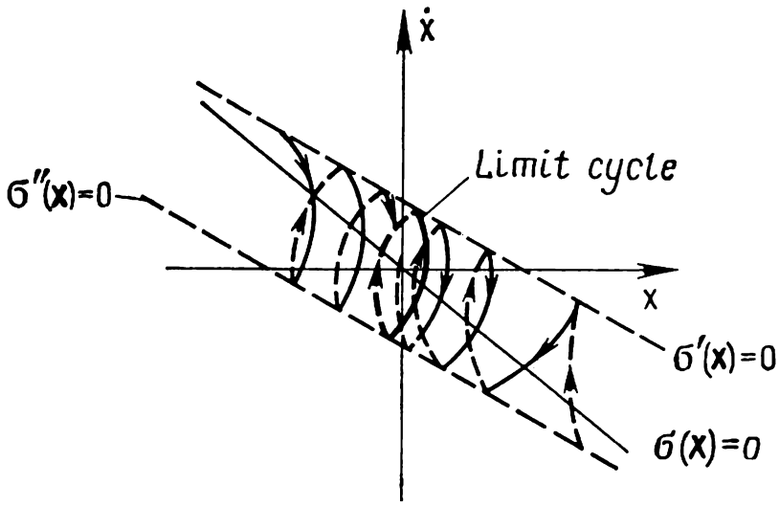

Figure 5.7

elements unaccounted for in the mathematical model (4.1), provided that these imperfections and inaccuracies in plant modeling approximation are sufficiently small [92, 94, 95].

Inspection of Fig. 5.7 suggests that two new features appeared in controlled process behavior as compared with that of the system with an ideal two-position element. The first feature is that now the controlled processes $x(t)$, generally speaking, fail to converge to O_x as $t \to \infty$. The second feature is that there is no such exponential estimate for the norms of controlled processes as has been used for the control system with an ideal on-off element from the moment the sliding mode occurs.

Summing up we may now conclude that large negative feedback fails to solve the control problem asymptotically, i.e. $\| x(t) \| \not\to 0$ as $t \to \infty$, when there are constraints imposed on the magnitude of control signal, and a variety of imperfections and approximation inaccuracies are taken into account. In the circumstances we may say only that the controlled processes become stable in some neighborhood $\| x \| \leqslant \Delta$ (Δ = constant > 0) of O_x, the size of this neighborhood depending on the $\mathcal{P}_x$ parameters, imperfections, and bounds on control so that there is no arbitrary choosing of this neighborhood. Exponential estimates for the norms of controlled processes, belonging to the considered class of processes, cannot be warranted, but we may assert that the domain $\| x \| \leqslant \Delta$ is stable, i.e. inequality (4.4a) is satisfied.

Denoting by $\mathcal{F}_{\text{pc}}$ the set of piecewise continuous time functions, and by $\mathcal{A}_\tau$ the control algorithm (5.5) now accounting for imperfections leads us to shorthand summarizing of the aforementioned characteristics of the respective closed-loop control systems as follows:

$$\mathcal{P}_x \in \mathcal{C}_{(\Delta,\, \mathcal{A}_\tau)} \{\mathcal{F}_{\text{pc}},\ I_2;\ N\}$$

5.5 Linear Zone

In this section we shall be concerned with qualitative evaluation of controlled processes behavior in the $\mathcal{P}_x$ control systems using $\mathcal{A}_k$ control algorithms with finite gain k and subject to a constraint of (5.4) type on the control signal. We recall that the limit situation that occurs in such control systems at $k = \infty$ has been treated under various assumptions in Sections 5.3 and 5.4. Here we shall assume that $k \leqslant k^0$, where k^0 is a constant defining the largest permissible gain in the $\mathcal{A}_k$-algorithm.

In the situation considered there exists a linear zone, or a range of σ values where the effect of constraint on the control is not observed and, therefore, the motion is steered by a linear control. From (5.4) it follows that this linear zone is defined by the inequality $|\sigma| < M/k$, and since $\sigma = \sigma(x)$ then on the $(x, \dot{x})$ plane to such a linear zone there corresponds a band of width $2M/k$ symmetrically spaced on either side of the line $\sigma(x) = 0$. For convenience, we shall denote the set of points belonging to this band on the phase plane by G_k, and assume in our subsequent considerations that the $\mathcal{P}_x$ parameters are fixed.

The behavior of phase trajectories on G_k is conditioned totally by the type of the quiescent point O_x for the control system on hand. Assume, for definiteness, that the limitation $k \leqslant k^0$ allows a choice of k in the $\mathcal{A}_k$ control algorithm such that for $x \in G_k$ the respective phase trajectories coincide with the trajectories plotted in Fig. 5.2. Then for any fixed values of the $\mathcal{P}_x$ parameters a_1 and a_2 from the range (4.2), within the set G_k there exist, in general, two asymptotes $c_k^i \dot{x} + x = 0$, $i = 1, 2$, the coefficients c_k^i, $i = 1, 2$, defining the position of these straight lines on the phase plane. It is essential that these coefficients depend on the $\mathcal{P}_x$ parameters and gain k, i.e., $c_k^i = c_k^i(a_1, a_2, k)$, and if $c_k^1 > c_k^2$, then in the limit of $k \to \infty$ we have $\lim c_k^1 = c$ and $\lim c_k^2 = 0$. For a finite k, the asymptotes intersect the boundaries $\sigma(x) = M/k$ and $\sigma(x) = -M/k$ of G_k in finite points, namely the first asymptote at points 2 and 2′, and the second at points 3 and 3′.

It will be noted that if $\mathcal{M}_k$ is the set of points of the line section [2, 2′], then in the limit as $k \to \infty$, $\mathcal{M}_k$ becomes $\mathcal{M}_\infty$ the size of which depends on the values of a_1 and a_2, the $\mathcal{A}_k$-algorithm degenerates to an $\mathcal{A}_{b\text{-}b}$ algorithm elucidated in Chapter 6, the set G_k contracts to the line $\sigma(x) = 0$ and therefore

$$\min_{a_1, a_2} \mathcal{M}_\infty = \mathcal{M}$$

where $\mathcal{M}$ is defined in Section 5.3.

Beyond the set G_k, the motion in the considered control system is effected by a constant control action equal in value to either M

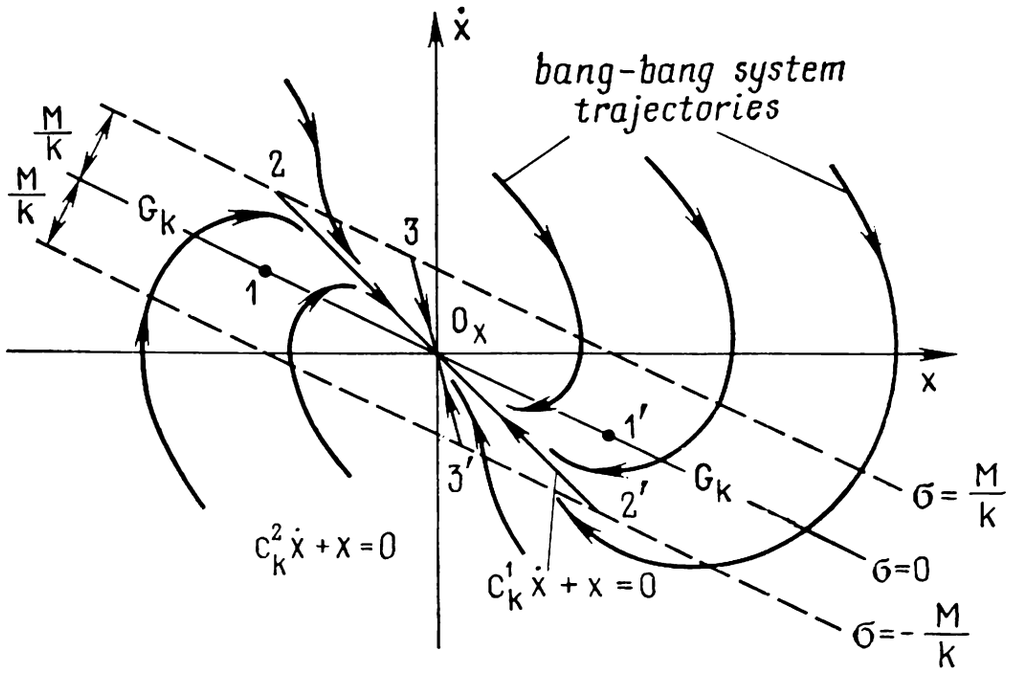

Figure 5.8

or $-M$. Therefore, the trajectories of this motion coincide with that of the bang-bang control system (see Fig. 5.5). From this consideration it follows that for some fixed values of the $\mathcal{P}_x$ parameters the phase portrait for the control system under examination will be as shown in Fig. 5.8.

Inspection of Fig. 5.8 suggests that such control systems are capable of yielding an asymptotical and stable solution to the control problem. The dynamic behavior of the controlled processes depends, however, on the $\mathcal{P}_x$ parameters at all stages of the motion, i.e., both for $x(t)$ inside and outside G_k, and this dependence becomes stronger as the gain in the $\mathcal{A}_k$-algorithm diminishes. Also, with a wide parameter-variation range (4.2) the asymptote $c_k^1 \dot{x} + x = 0$ can, in general, adopt any position within the set G_k, and it is rather tedious to derive an exponential estimate for the norms of controlled processes even for motion within G_k.

Denoting by $\mathcal{F}_c$ the set of functions continuous on $[t_0, \infty)$, and by $\mathcal{A}_k$ the linear control algorithm accounting for the constraint (5.4), we may summarize the aforementioned qualitative conclusions in shorthand as follows:

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_k}\{\mathcal{F}_c,\ I_2,\ N\}$$

Of course, the dependence of controlled process behavior on the plant parameters in such a control system can be decreased by increasing the gain k whenever possible. In this case the set G_k is contracting to the set $\sigma(x) = 0$ as $k \to \infty$ and beginning from some value of k the actual system behavior will be markedly affected by

dynamic elements and other nonlinearities unaccounted for in the plant model. As the gain increases, the controlled processes in the real system become at first oscillatory then go unstable. For sufficiently large gains the behavior of this control system will be close to that of the control system investigated in Section 5.4.

Consequently, the control algorithm considered in this section also fails to solve the control problem formulated in Chapter 4.

5.6 Coordinate Feedback Under Inaccurate Information

This section will evaluate the effect of inaccuracies in deriving information about the $\mathcal{P}_x$-process status on the stability of the associated control system using $\mathcal{A}_k$-algorithms for control. We shall assume that the constraint on the control signal $u(t)$ is lifted, but there exists a limitation from above imposed on the modulus of gain k in the $\mathcal{A}_k$-algorithm, that is in (5.1) $0 \leqslant |k| \leqslant k^0$, where k^0 is a finite constant. The main purpose of this evaluation will be to determine the nature of constraints on the class of $\mathcal{P}_x$-processes such that can be steady controlled.

Consider a family of $\mathcal{P}_x$-processes for which the parameters a_1 and a_2 are fixed but can take on any values within (4.2). Let, for the moment, assume that at every time instant $t \geqslant t_0$ the information about each $\mathcal{P}_x$-process is available as a pair of functions $[x(t)\ \dot{x}(t)]$. Then the $\mathcal{A}_k$-algorithm (5.1) applied to control yields a closed-loop control system with the characteristic equation

$$s^2 + (kc + a_2)\, s + (k + a_1) = 0$$

When k is chosen to satisfy the conditions

$$kc > -a_2 \quad \text{and} \quad k + a_1 > 0 \tag{5.8}$$

which must be met for all pairs (a_1, a_2) from (4.2), the closed-loop control system will be asymptotically stable for any $\mathcal{P}_x$-process from the considered family. We recognize that these inequalities may not be met because of the constraint $|k| < k^0$. Let this not be the case, i.e., $k^0 > \max(-a_1, -a_2/c)$ for all a_1 and a_2 from (4.2).

Assume now that $\dot{x}(t)$ is not known exactly and only its estimate $\hat{x}(t)$, $t \geqslant t_0$ can be used for control. Let, for definiteness, this estimate $\hat{x}(t)$ be the solution to

$$T\dot{\hat{x}}(t) + \hat{x}(t) = \dot{x}(t), \quad t \geqslant t_0$$

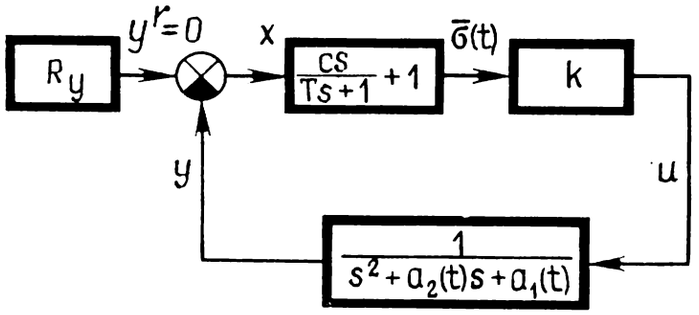

Figure 5.9

where T is a positive constant. We wish to examine how the substitution of $\hat{x}(t)$ in the $\mathcal{A}_k$ control algorithm for $\dot{x}(t)$ will affect stability of the control system on hand.

The algorithm of control with the imperfect derivative will be denoted by $\mathcal{A}_{k,T}$, and the information used for this purpose by $\hat{I}_2 = \{x(t), \hat{x}(t), t \geqslant t_0; a_i^{\pm}, i = 1, 2\}$. The algorithm $\mathcal{A}_{k,T}$ proper is defined as

$$u(t) = k\bar{\sigma}(t) \tag{5.9}$$

$$\bar{\sigma}(t) = c\hat{x}(t) + x(t), \quad t \geqslant t_0$$

The block diagram corresponding to the closed-loop control system is depicted in Fig. 5.9.

Under the above assumptions, for any $\mathcal{P}_x$-process the behavior of such a control system will be completely defined by its characteristic equation

$$Ts^3 + (Ta_2 + 1)s^2 + (Ta_1 + Tk + a_2 + kc)s + (k + a_1) = 0$$

Given inequalities (5.8) are satisfied, the stability conditions for this control system are defined by

$$Ta_2 + 1 > 0 \tag{5.10}$$

$$kr_1 > -a_2r_2$$

where for convenience we have denoted

$$r_1 = T^2a_2 + Ta_2c + c \quad \text{and} \quad r_2 = T^2a_1 + Ta_2 + 1$$

The constraints introduced by (5.10) in addition to those of (5.8) narrow the family of $\mathcal{P}_x$-processes for which stability can be insured under inaccurate information on the system status.

A qualitative insight into how essential these constraints are may be obtained from comparison of the plots in Fig. 5.10*a* and *b*. In the first of them, G_k represents the area of stable systems with $\mathcal{A}_k$ control algorithms, while in the second diagram $G_{k,T}$ represents the stable systems with $\mathcal{A}_{k,T}$ control algorithms. The area G_T, cross-hatched in two directions, represents the $\mathcal{P}_x$-processes for which

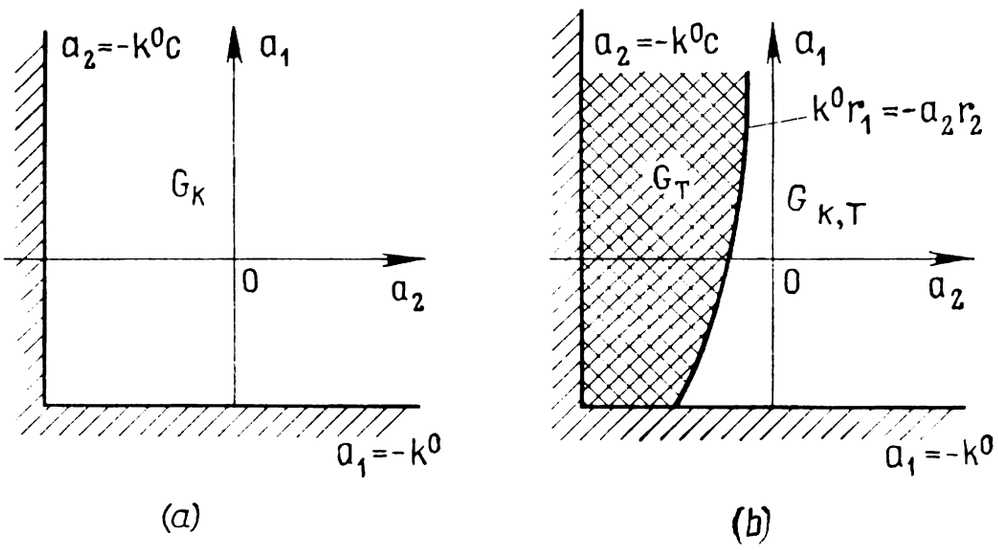

Figure 5.10

no stable control can be secured by $\mathcal{A}_{k,t}$ -algorithms with limited gain. We note that both diagrams are for the case of $Tk^0c < 1$.

Thus, the use of linear $\mathcal{A}_k$ control algorithms with limited gain in the conditions of inaccurate information on the process status may lead to the loss of stability in this closed-loop control system. Of course, the general character of this finding cannot change when the $\mathcal{P}_x$ parameters arbitrarily vary within (4.2).

We will demonstrate below that algorithms preserving process stability in the closed-loop system under the aforestated conditions and constraints can be derived in the class of binary automatic-control systems. The stability of controlled processes in such systems will be achieved not only for any parameters $(a_1, a_2) \in G_k$ but also for any mode of their time variation within the set G_k. Moreover, aperiodic behavior of transient processes will be ensured.

5.7 Inaccuracies in Model Approximation

In this section we shall touch upon the topics close to those discussed in Section 5.6. More specifically, we shall seek to establish the nature of constraints on the class of $\mathcal{P}_x$-processes which can be steady-controlled with $\mathcal{A}_k$ limited-gain algorithms with due account of dynamic nonlinearities. It is assumed that there is no limitation on the control signal $u(t)$.

In building a mathematical model of a process to be controlled the analyst has to neglect many factors including so-called small parameters which within some limits and under certain conditions have no effect on the steady-state and dynamic behavior of the open-loop dynamic system. A feedback loop introduced into the system may cause the unaccounted parameters to play a significant role and markedly distort the behavior of the controlled processes.

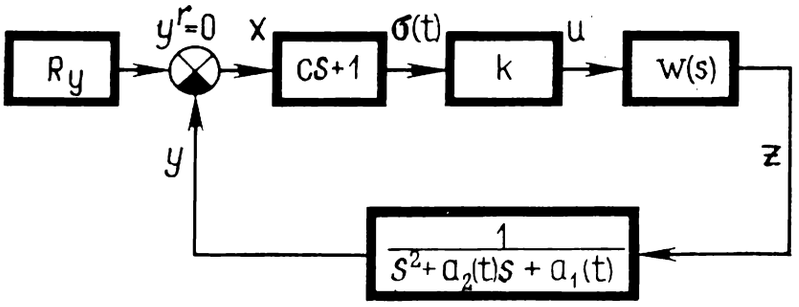

Figure 5.11

Let us look at the situation with the small parameters being constant and associated with some element of transfer function $W(s)$ embedded into the block diagram of the system with an $\mathcal{A}_k$-algorithm as indicated in Fig. 5.11. Here $z(t)$ is the result of conversion of the control signal $u(t)$ by this dynamic element $W(s)$. It will be noted that the operator $W(s)$ can represent not only the inaccuracies of approximation, but also the system actuator or the inertia of measurement devices forming the function $\sigma(t)$. We will be interested here in the case where $W(s)$ is an inertial element with a time constant $T > 0$, i.e., $z(t)$ and $u(t)$ are related by the equation

$$T\dot{z}(t) + z(t) = u(t), \quad t \geqslant t_0 \tag{5.11}$$

The characteristic equation—the parameters a_1 and a_2 are assumed here, as in Section 5.6, to be constant—of this closed-loop control system has the form

$$Ts^3 + (Ta_2 + 1)\, s^2 + (kc + Ta_1 + a_2)\, s + k + a_1 = 0 \tag{5.12}$$

At $T = 0$ the control system is seen to be stable, given (5.8) is satisfied. With the constraint $0 \leqslant |k| \leqslant k^0$, this again implies that $\mathcal{A}_k$-algorithms are helpful for steady controlling all $\mathcal{P}_x$-processes with the parameters (a_1, a_2) from the G_k area indicated in Fig. 5.10*a*. For $T > 0$, the stability conditions are also defined by

$$\begin{aligned} &Ta_2 + 1 > 0 \\ &Ta_2 + kc + a_2 > 0 \\ &kr_1' > -a_2 r_2' \end{aligned} \tag{5.13}$$

where $r_1' = Tca_2 + c - T$ and $r_2' = 1 + Ta_2 + T^2 a_1$.

The area $G_{k,T}'$ representing the $\mathcal{P}_x$-processes amenable to stable control in this case subject to $Tk^0 c < 1$ and $T < c$ is displayed in Fig. 5.12. Two-directional crosshatching again emphasizes the area, G_T', which appears owing to the constraint imposed on the gain of the $\mathcal{A}_k$-algorithm involved.

Thus, the bound from above imposed on the $\mathcal{A}_k$-algorithm gain may considerably reduce the class of $\mathcal{P}_x$-processes for which stable control is feasible in this case.

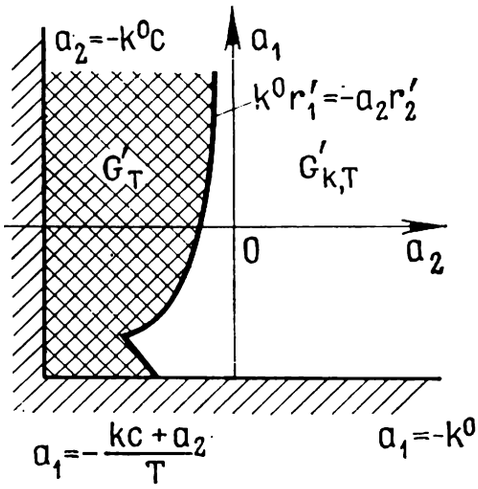

Figure 5.12

Below we shall demonstrate that the effect of the aforementioned constraints can be markedly relaxed in the class of binary control systems. We shall construct control systems with aperiodic transients for arbitrary behavior of $\mathcal{P}_x$ parameters in some subset of G_k.

Simultaneous account of inaccuracies in taking derivatives and model approximation—which is a typical situation in practice—can further reduce the class of $\mathcal{P}_x$-processes amenable to stable control with $\mathcal{A}_k$ limited-gain algorithms. This leads us to the following qualitative conclusion. Limitations imposed on gain and amount of available information severely constrain the practical capabilities of linear feedback in ill-defined dynamic systems. Therefore, transfer to nonlinear control algorithms seems to be justified. The algorithms of binary control systems will be shown to constitute a subclass of nonlinear algorithms.

6 Coordinate-Operator Feedback

In this chapter we are content to study the possibilities of coordinate-operator feedback (COFB) for solving the control problem formulated in Chapter 4. The generalized block diagram associated with control systems considered in this chapter is pictured in Fig. 6.1. A specific control system is defined by specifying the operators R_x, C_u, and C_μ. All algorithms consistent with this block diagram will be denoted by $\mathcal{A}_\mu(q)$, where q gives the number of components in the system's state vector $x(t)$ used in the coordinate feedback (CFB) law. We assume, for definiteness, that at $q = 1$, writing $\mathcal{A}_\mu(1)$ implies that the CFB law involves the first component of x, i.e., $u(x, t) = u(x_1, t)$; and at $q = 2$, the two leading components of x are used, $u(x, t) = u(x_1, x_2, t)$, and so on.

For brevity, any control system using an $\mathcal{A}_\mu(q)$-algorithm for $\mathcal{P}_x$-process (respectively S-system) will be denoted $S_\mu(q)$ and referred

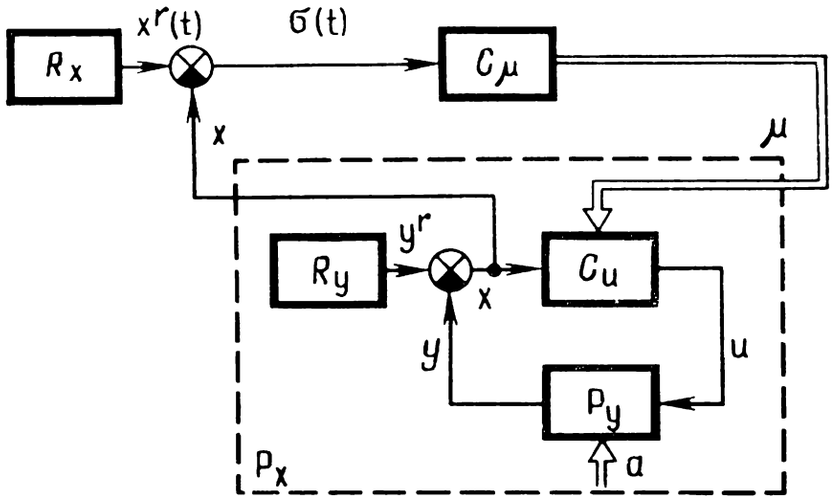

Figure 6.1

to as an S_μ (q) system, or simply as an S_μ system when the value of q is evident from the context.

Let us now define more accurately what in this text is implied by the *complete solution of the control problem.* We are intent to formulate this concept first in words to refine it then in a more formal way.

From the preceding consideration it follows that for many cases of handling dynamic systems with uncertain parameters (we will refer to such systems as ill-defined or uncertain) those control algorithms are of most practical significance which exhibit or offer the following properties:

— use limited gain in CFB loops; this means, specifically, a diminished effect of constraints on the control signals in the CFB loop;

— generate control signals as a continuous function of time; this facilitates operation of certain actuators;

— ensure stable control of ill-defined processes and a warranted settling time;

— allow a weak dependence of plant dynamics in the closed-loop control system upon widely variable plant characteristics;

— retain their key properties in the presence of inaccuracies in model approximation and data collection.

Some of the aforelisted features need, of course, be refined. For the problem of $\mathcal{P}_x$-process control with parameters from (4.2), the respective refinements are given in the following definition.

We will say that an $\mathcal{A}$-algorithm provides the *complete solution to the problem of ill-defined* $\mathcal{P}_x$*-process control* if

(1) for all $t \geqslant t_0$ the control function $u\ (x,\ t)$ satisfies one of the following inequalities:

$$| u\ (x,\ t)\ | \leqslant k_x^1\ \| x\ \| \tag{6.1a}$$

$$| u\ (x,\ t)\ | \leqslant k_x^1\ \| x\ \| + k_x^2 \tag{6.1b}$$

where k_x^i are some positive constants such that $k_x^i < \infty$, $i = 1, 2$;

(2) for any fixed $t \geqslant t_0$ and for any pair of vectors x' and x'' the inequality

$$| u(x', t) - u(x'', t) | \leqslant L_x \| x' - x'' \| \tag{6.1c}$$

is satisfied for some positive constant $L_x < \infty$, or holds outside some neighborhood of the phase space origin;

(3) for each solution of the closed-loop control system the function $u(t) = u[x(t), t]$ is continuous for all $t \geqslant t_0$;

(4) in the closed-loop control system, every controlled process $x(t)$, beginning with a certain finite time instant $t_x \geqslant t_0$, possibly dependent on $x(t)$, at certain positive constant $\delta_0 > 0$ satisfies the differential equation

$$\dot{x}(t) + \delta_0 x(t) = \chi(t) \tag{6.2a}$$

where $\chi(t)$ is a function of time such that for $t \geqslant t_x$

$$| \chi(t) | \leqslant \delta(t) | x(t) | \tag{6.2b}$$

Here $\delta(t)$ is a nonnegative time function bounded for all $t \geqslant t_x$, viz.

$$0 \leqslant \delta(t) \leqslant \delta_1 < \delta_0 \tag{6.2c}$$

If the constant δ_1 is small compared to δ_0 in some sense, then we will say in the same sense of *weak dependence* of controlled process dynamics on $\mathcal{P}_x$ parameters from t_x onwards (independence accurate to δ_1). If $\delta(t) \to 0$ as $t \to \infty$, then we say of *asymptotic independence* of $x(t)$ dynamics on $\mathcal{P}_x$ parameters. Finally, if $\delta(t) \equiv 0$, then we say of *total independence* of $x(t)$ dynamics of $\mathcal{P}_x$ parameters beginning from time t_x. We shall strive to build a control algorithm which ensures the complete solution of this control problem.

In what follows δ_0 is assumed to be specified and equal to $1/c$, where c is the same constant that appears in Chapter 5. With this assumption, the shorthand for items (3) and (4) from the above definitions, with the nomenclature as given in Chapter 4, is as follows:

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}} \{\mathcal{F}_c, I; N, 1/c - \delta_1\} \tag{6.3a}$$

For $\delta_1 \ll 1/c$, this expression implies weak dependence of $x(t)$ on the parameters $a_1(t)$ and $a_2(t)$. An alternative shorthand expression

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}} \{\mathcal{F}_c I; N, 1/c\} \tag{6.3b}$$

implies a total independence of $x(t)$ of $a_1(t)$ and $a_2(t)$. In both expressions (6.3) $\mathcal{F}_c$ stands for a class of functions continuous for $t \geqslant t_0$.

Now we embark on constructing a binary control system with coordinate-operator feedback (COFB), Fig. 6.1. Define at first the

type of reference input operator R_x, proceeding as outlined in Ref. 88. It is quite obvious that with the signal of the reference input element R_x taken in the form

$$x^r(t) = -c\dot{x}(t) \tag{6.4}$$

for all $t \geqslant t_0$, the COFB loop error $\sigma(t)$ will be given for all t as

$$\sigma(t) = -[c\dot{x}(t) + x(t)]$$

Assume that beginning from a certain time instant the equation $\sigma(t) = 0$ is satisfied for a certain controlled process $x(t)$, then for this process

$$\dot{x}(t) = -\frac{1}{c}x(t)$$

and (6.3b) holds true. Therefore it would be appropriate to define the operator R_x as $-c\mathrm{d}(\cdot)/\mathrm{d}t$. This form of the operator is a general one for all $\mathscr{A}_\mu$ control algorithms involved in the ensuing consideration.

Our objective now is to select operators C_u, C_μ and their parameters such that to ensure for each controlled process $x(t)$ the equality $\sigma(t) = 0$ from some moment of time. We note from the very outset that (6.3b) does not hold for the class of $\mathscr{A}_\mu$ control algorithms, but for some of $\mathscr{A}_\mu$-algorithms an approximation, however close, can be achieved as represented by (6.3a) with a sufficiently small δ_1.

In this section we evaluate various $\mathscr{A}_\mu$-algorithms for control of $\mathscr{P}_x$-processes with parameters within (4.2), investigate in depth the behavior of controlled processes in the respective S_μ systems, and establish to what extent a specific $\mathscr{A}_\mu$ control algorithm suits achieving the complete solution of the control problem. Especial attention will be drawn to smooth $\mathscr{A}_\mu$-algorithms whose instrumentation with digital elements is well standardized.

$\mathscr{A}_\mu$ (1)-algorithms will be considered first, and the discussion of $\mathscr{A}_\mu$ (2)-algorithms is reserved for later sections in the chapter. Considering $\mathscr{A}_\mu$ (1)-algorithms we assume that for each fixed μ the CFB-loop operator C_u performs the transformation

$$u = k(\mu)x \tag{6.5}$$

which is seen to be linear in x with a coefficient $k(\mu)$ being a linear function of μ (with some constant $\bar{k}$ individual for each $\mathscr{A}_\mu$-algorithm)

$$k(\mu) = \bar{k}\mu \tag{6.6}$$

The specific features of each $\mathscr{A}_\mu$ (1)-algorithm and, consequently, the associated S_μ system are governed by the law prescribing the generation of variable μ.

6.1 Proportional Coordinate-Operator Feedback

Consider first the situation with the operator C_μ realizing for a positive constant α the function

$$\mu = -\alpha\sigma \tag{6.7}$$

The S_μ system that results may be represented by one of the two block diagrams, shown in Fig. 6.2, differing in the way they realize (6.5) and (6.6). In the block diagram at (*a*) these dependences are realized by changing the CFB law parameters; accordingly in a number of publications [84-88] we referred to this method of altering the CFB law as coordinate-parametric.

The other block diagram at (*b*) exploits a multiplicative variation of the CFB law which corresponds to the coordinate-operator action. Because the difference between these diagrams is purely technological in what follows we shall employ both of them.

We recall that the ultimate goal of $\mathscr{A}_\mu$-algorithm implementation is to provide conditions such that for each x (t) starting from some time instant the equality $\sigma(t) = -[c\dot{x}(t) + x(t)] = 0$ holds.

In accord with (6.4)-(6.6), for the $\mathscr{P}_x$-process

$$\dot{\sigma} = c\bar{k}x\mu + ca_1x + (ca_2 - 1)\dot{x} \tag{6.8}$$

The quantity μ is the control of the COFB loop that is devised to ensure the equality $\sigma(t) = 0$. In (6.8) this quantity, however, is preceded by a factor which changes sign with x. Consequently, a change of sign for x implies a sign reversal for feedback in the COFB loop. Hence, the law (6.7) cannot in principle ensure the required equality as it fails to keep a constant sign of feedback in the COFB loop. It is clear that negative feedback in such a loop will be effected if in place of (6.7) we invoke a law taking care of sign reversal of the factor of μ in (6.8). This can be achieved, for instance, with the following modification of (6.8):

$$\mu = -\alpha\sigma \operatorname{sgn} x = \bar{\mu} \operatorname{sgn} x \tag{6.9}$$

Indeed, substituting (6.9) in (6.8) we get

$$\dot{\sigma} = -\alpha c\bar{k} \mid x \mid \sigma + ca_1x + (ca_1 - 1)\dot{x} \tag{6.10}$$

Here the factor of σ is nonpositive.

The block diagram of S_μ system resulting with this COFB law is presented in Fig. 6.3. The output of the on-off element sgn x is indicated by a double-shafted arrow as it is this signal that defines the transformation of the signal $\bar{\mu}(t)$ into $\mu(t)$ and therefore, in accord with the above definitions, is an operator signal.

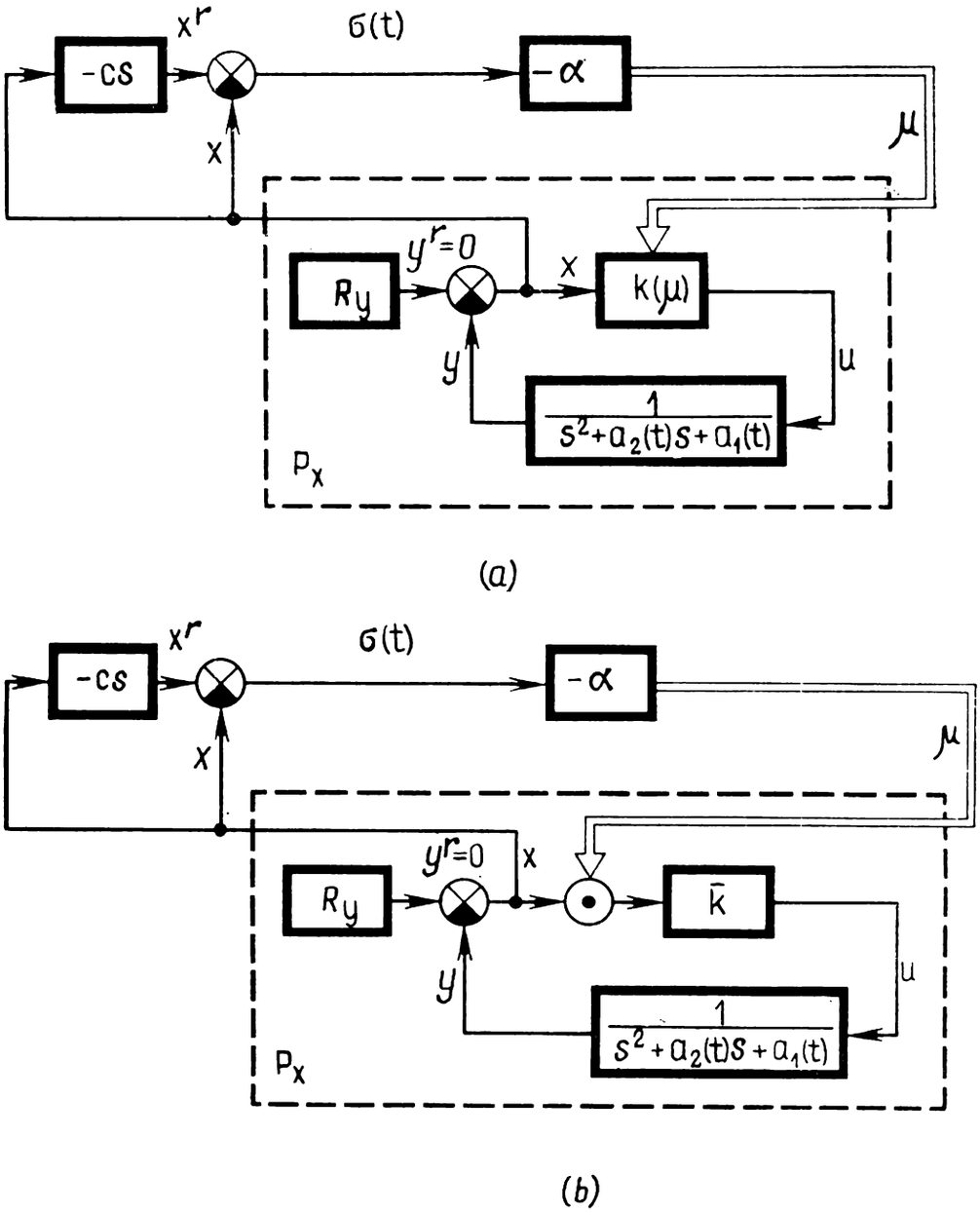

Figure 6.2

It is worth emphasizing that the need for nonlinear control laws (6.9) in the COFB loop is of principal significance and is predetermined by the nonlinear behavior of such feedback. This feature of coordinate-operator feedback was outlined in the overview of Part One.

Turning again to the block diagram of $S_\mu$system in Fig. 6.3, we would like to look at system's equilibrium positions, denoted below as $(x^*, 0)$. Assuming the parameter $a_1(t)$ being constant for all $t \geqslant t_0$, from the equation of $\mathcal{P}_x$-process and equations (6.5), (6.6) and (6.9) we find that all the equilibrium points of the S_μ system under consideration are the roots of the equation

$$a_1 x^* = -\alpha \bar{k} x^* \mid x^* \mid \tag{6.11}$$

at $\dot{x}^* = 0$.

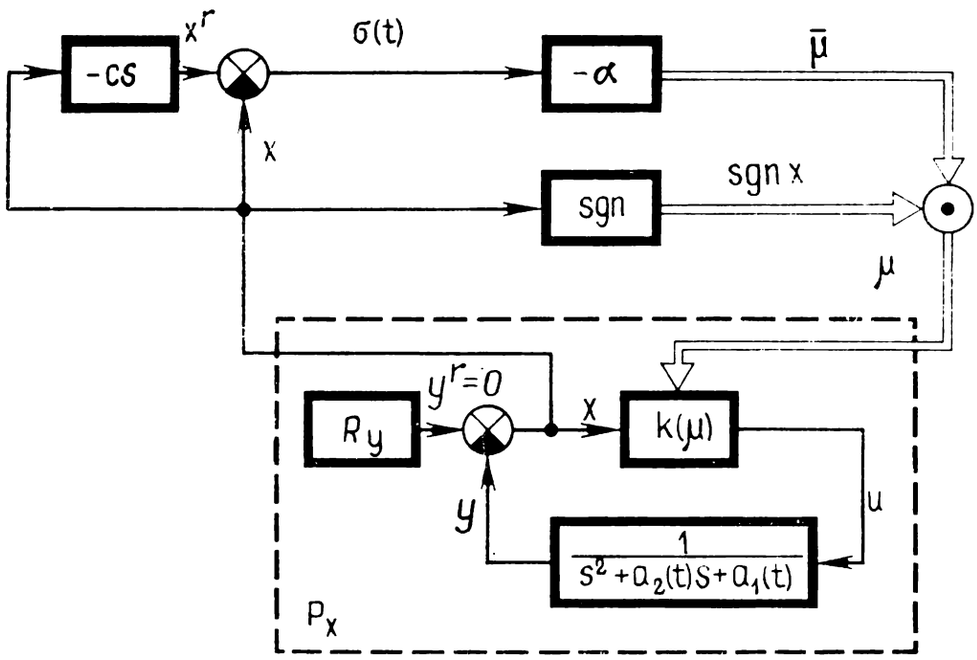

Figure 6.3

If $a_1 < 0$ and this inequality is allowed by the constraints (4.2), then such a control system may have three equilibrium positions defined at $\dot{x}^* = 0$ as $x_1^* = 0$, $x_{2,3} = \pm(a_1/\alpha\bar{k})$, of which only the first is desirable.

From inspection of the closed-loop control system equation

$$\ddot{x} + (a_2 + \alpha c\bar{k} \mid x \mid)\dot{x} + (a_1 + \alpha\bar{k} \mid x \mid)x = 0$$

it is not hard to see that under the assumption made the equilibrium position of $x_1^* = 0$ is unstable, while that of $\mid x_{2,3}^* \mid = -(a_1/\alpha\bar{k})$ are stable in the small when a_2 is constant for $t \geqslant t_0$ and $a_2 - ca_1 > 0$. Therefore, the controlled processes in such a control system may exhibit a rather complex behavior. This implies that the aforestated control problem cannot be solved with the static COFB (6.9). There is a need for securing stability for the point $(x_1^*, 0)$ and eliminate the influence of other equilibrium points different from the origin O_x. This need can be handled by various means, some of which will be covered later. For the time being we confine ourselves to the effects that occur in S_μ systems when constraints are imposed in the COFB loop.

6.2 Proportional Coordinate-Operator Feedback with Magnitude Bounded Output

Suppose that in solving the control problem under the aforestated assumptions we may limit ourselves to achieving the property of the process being steady quasicontrolled. We shall de

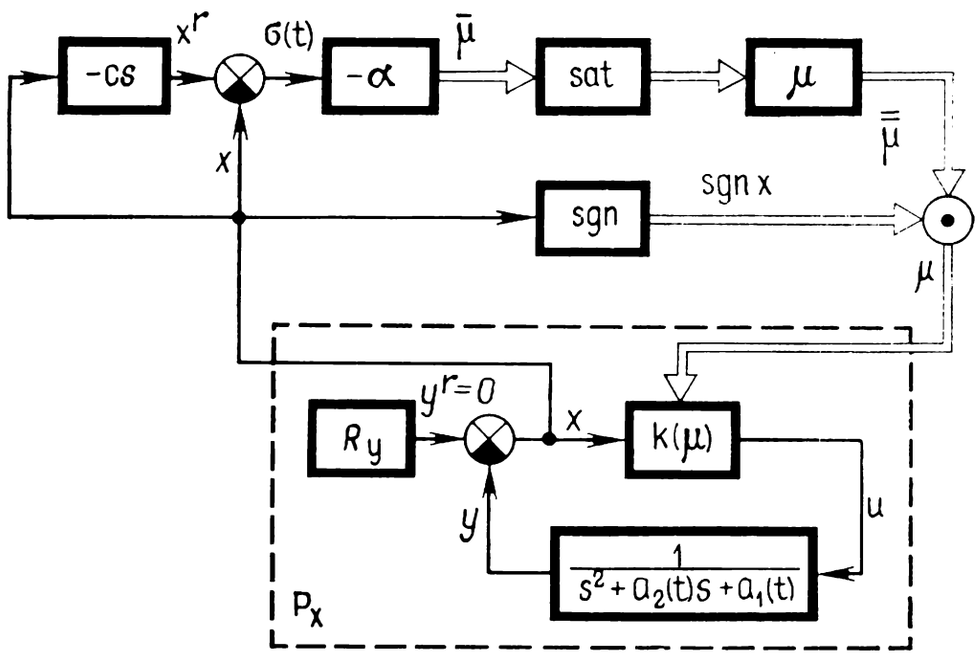

Figure 6.4

monstrate that this is attainable by introducing a limitation in the COFB loop.

Consider a function sat (λ) of scalar argument λ such that sat $(\lambda) = 1$ for $\lambda > 1$

$$= \lambda \quad \text{for} \quad |\lambda| \leqslant 1 \qquad (6.12)$$
$$= -1 \quad \text{for} \quad \lambda < -1$$

and introduce the constraint it offers in the COFB loop of the control system examined in Section 6.1. This being the case, the law (6.9) must be replaced with the COFB law of the form

$$\mu = \bar{\mu} \operatorname{sgn} x = \mu_0 \operatorname{sat} (\bar{\mu}) \operatorname{sgn} x$$

$$\bar{\mu} = -\alpha\sigma \qquad (6.13)$$

μ_0 being a positive constant. The block diagram of this S_μ system assumes the form depicted in Fig. 6.4. The set (6.13) allows for sat $(-\lambda) = -\text{sat}\,(\lambda)$.

For this S_μ system, the phase plane $(x, \dot{x})$ naturally partitions into three areas (Fig. 6.5), namely,

$$G_\eta = \{x: \ |\sigma(x)| \leqslant \eta, \quad \eta = 1/\alpha\}$$
$$G_\eta^+ = \{x: \ \sigma(x) > \eta\} \qquad (6.14)$$
$$G_\eta^- = \{x: \ \sigma(x) < -\eta\}$$

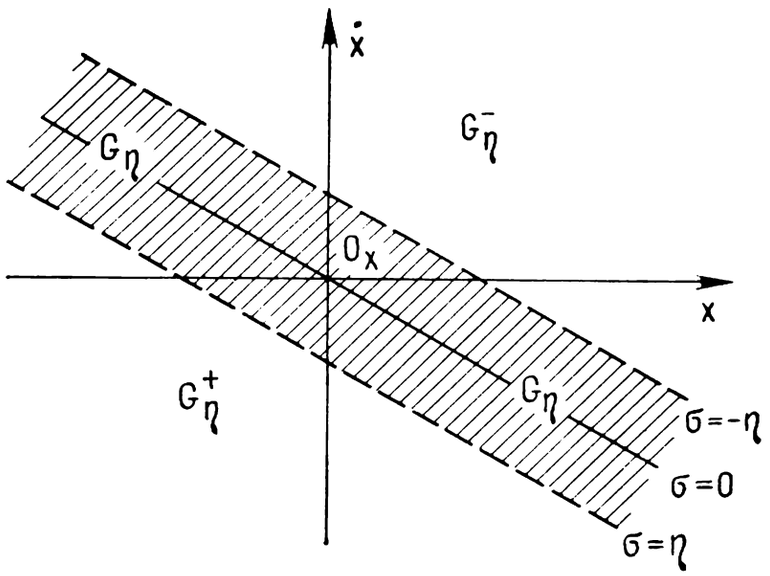

Figure 6.5

in each of which the behavior of the control system is governed by an individual equation of motion. From inspection of Fig. 6.4 it is an easy matter to see that these equations are

$$\begin{aligned} \ddot{x}+a_2\dot{x}+a_1x &= \bar{k}\mu_0\alpha\sigma\,|x| && \text{for } \boldsymbol{x}\in G_\eta \\ &= \bar{k}\mu_0\,|x| && \text{for } \boldsymbol{x}\in G_\eta^+ \\ &= -\bar{k}\mu_0\,|x| && \text{for } \boldsymbol{x}\in G_\eta^- \end{aligned} \tag{6.15}$$

Let us provide a deeper insight into the properties of the controlled processes in such a control system. Let for the moment $\boldsymbol{x}(t_0)\notin G_\eta$ and, for definiteness, $\boldsymbol{x}(t_0)\in G_\eta^-$; the case of $\boldsymbol{x}(t_0)\in G_\eta^+$ can be examined in a similar manner. If $\boldsymbol{x}(t_0)\in G_\eta^-\cap\{\boldsymbol{x}\colon x<0\}$, then it can be readily verified, for example by inspecting Fig. 6.5, that the respective controlled process in a finite time interval either reaches the boundary of the set G_η or leaves the area $G_\eta^-\cap\{\boldsymbol{x}\colon x\leqslant 0\}$. If now $\boldsymbol{x}(t_0)\in G_\eta^-\cap\{\boldsymbol{x}\colon x>0\}$, then from (6.15) for $t\geqslant t_0$ we obtain the equation

$$\ddot{x}+a_2\dot{x}+(a_1+\bar{k}\mu_0)\,x=0 \tag{6.16}$$

satisfied by such controlled process.

It has been demonstrated [24, 25] that in order that an arbitrary solution to equation (6.16) appears in the set G_0 (and, consequently, on the boundary of the set G_η), it is sufficient that at any fixed parameters a_1 and a_2 from (4.2) the respective characteristic equations derived from (6.16) have no positive roots. This will take place, for example when

$$\bar{k}\mu_0>\sup_{t\geqslant t_0}\left[\frac{a_2^2(t)}{4}-a_1(t)\right] \tag{6.17}$$

Below we also use other conditions warranting the arrival of the controlled processes at the boundary of set G_η when driven from the areas $G_\eta^{\pm}$, namely,

$$c\bar{k}\mu_0 > \sup_{t \geqslant t_0} \left[ca_1(t) - \frac{ca_2(t) - 1}{c} \right] \tag{6.18a}$$

$$\inf_{t \geqslant t_0} ca_2(t) > 1 \tag{6.18b}$$

The validity of this immediately follows from analysis of an expression for $\dot{\sigma}(t)$ which for $x(t) \in G_\eta^-$ has the form

$$\dot{\sigma} = -\frac{ca_2 - 1}{c}\sigma + \left(ca_1 - \frac{ca_2 - 1}{c} \right) x + c\bar{k}\mu_0 |x| \tag{6.19}$$

It will be noted that the condition (6.18b) limits the class of $\mathcal{P}_x$-processes for which $\mathcal{A}_\mu$-algorithms of the type being discussed warrant the controlled processes be driven to the set G_η.

Thus, when (6.17) or (6.18) are satisfied, an arbitrary controlled process $x(t)$ is steered to the boundary of the set G_η in a finite time. In order to look at the further behavior of $x(t)$ we determine the derivative $\dot{\sigma}(t)$ at $\sigma = -\eta$. From (6.19) we get

$$\dot{\sigma} = \frac{ca_2 - 1}{c}\eta + \left(ca_1 - \frac{ca_2 - 1}{c} \right) x + c\bar{k}\mu_0 |x| \tag{6.20}$$

Observing in (6.20) the inequalities (6.18) we obtain that $\dot{\sigma}(t) > 0$ at $\sigma = -\eta$ (similarly, $\dot{\sigma}(t) < 0$ at $\sigma = \eta$). Consequently, for such $\mathcal{P}_x$-processes, $x(t)$ once driven into the set G_η never leaves this set at all subsequent t.

It will be worthwhile to note at this point, that for motion within the set G_η, the behavior of the control system on hand is described by the first differential equation in (6.15), which coincides with the dynamic equation of the control system from Section 6.1. Therefore, the case under consideration can also have three equilibrium points in G_η at a constant $a_1 < 0$ for $t \geqslant t_0$. This time, these points are defined by the roots of the equation

$$a_1 x^* = -\alpha\bar{k}\mu_0 x^* \mid x^* \mid$$

A suitable choice of the parameter μ_0 can, of course, drive nonzero equilibrium positions beyond the set G_η, but this method denies satisfying (6.18a). It is not hard to verify that when the inequalities (6.18) are met, the nonzero equilibrium positions belong to G_η and are stable in the small.

To derive some estimates of the norms of controlled processes subject to $x(t) \in G_\eta$ for all $t \geqslant t_1$, we may exploit the differential

relationship between $x(t)$ and $\sigma(t)$, called the differential constraint below, viz.,

$$c\dot{x} + x = -\sigma, \quad |\sigma(t)| \leqslant \eta, \quad t \geqslant t_1 \tag{6.21}$$

This leads us to the inequality $|x(t)| \leqslant]|x(t_1)| - \eta] \times \exp[-(t - t_1)/c] + \eta$ valid for all $t \geqslant t_1$. This inequality used with (6.18) enables us to establish that beginning from some time instant, each controlled process finds itself embedded into the sphere $\|x\| \leqslant \bar{\eta}$, where $\bar{\eta} = \eta(1 + 2/c)$ to remain therein at subsequent time.

Let us attempt now a systematic interpretation of the evidence we have acquired in the preceding paragraphs. The limitation inserted in the coordinate-operator feedback loop has considerably simplified the investigation of nonlinear system behavior. This constraint stabilizes all controlled processes in some neighborhood of the selected solution, the size of the neighborhood being selected at will as $\eta = 1/\alpha$, where α is the COFB loop gain. Because inequality (6.18) can be readily satisfied by a suitable choice of parameters and the validity of the bound $\|x(t)\| [< N \{\bar{\eta} + r[x(t_0)], U_{\bar{\eta}}\}$ for some N is evident, what we have said above implies that given (6.18b) is met the $\mathcal{P}_x$-process is steady quasicontrolled, or in shorthand notation $\mathcal{P}_x \in \mathcal{QC}_{\mathcal{A}_\mu}\{\mathcal{F}_c, I; N\}$.

This simplification of S_μ system analysis may be formally explained by the fact that beyond the set G_η, all S_μ system motions subject to the constraint (6.12) are described by piecewise linear differential equations, whereas inside G_η by nonlinear equations, but one may not analyze the latter confining himself to the use of the linear differential relationship (6.21) subject to a constraint on its right-hand side. Hence, introducing a limitation in the COFB loop allows us to confine ourselves to examining linear differential equations in the analysis of a nonlinear control system. For the sake of comparison, it is worth noting that limitations on the control signal introduced in the coordinate feedback loop, as a rule, degrade the control system behavior and impair investigation of such systems.

It will be in order to emphasize one more feature of system behavior which appears when there is a limitation on the control signal in the COFB loop. This limitation automatically satisfies the constraints on the CFB loops gains. Moreover, it allows an increase in the COFB loop gain, which produces a positive effect on the controlled process behavior, as the dependence on the $\mathcal{P}_x$ parameters weakens and the neighborhood of the origin O_x where they get stabilized also diminishes. That the dependence of $x(t)$ on the $\mathcal{P}_x$ parameters—understood, however, in a sense other than that defined in (6.2)—decreases may be concluded by comparing the error variation law in the S_μ

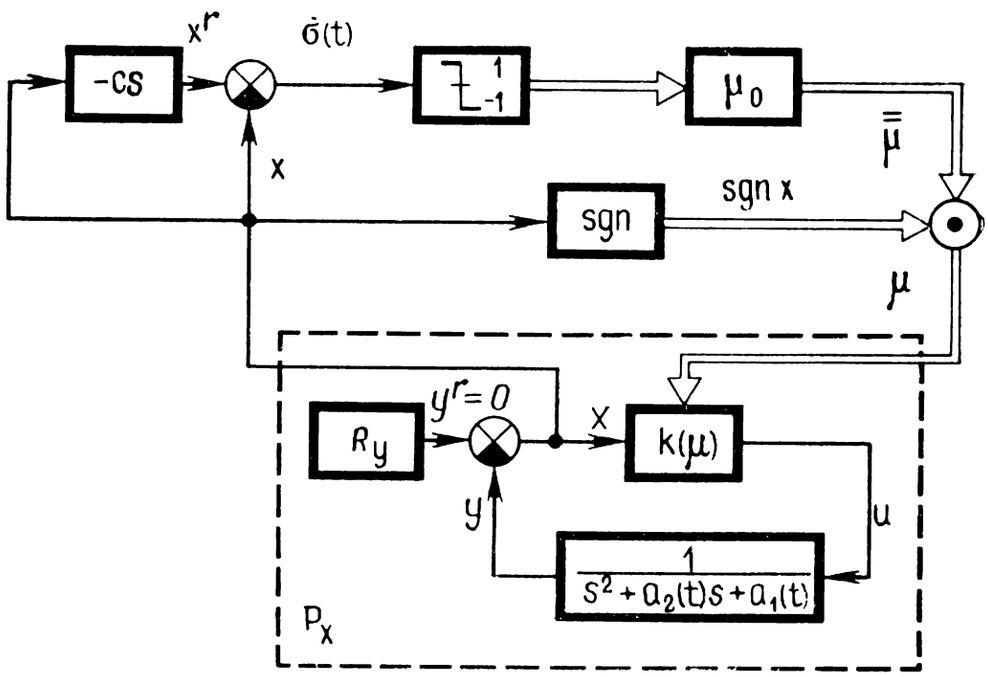

Figure 6.6

system under study with the error $\tilde{x}(t)$ for the ideal control system which manages in some way to keep the exact equality $\sigma(t) = 0$. In this case $\tilde{x}(t)$ varies in agreement with the equation

$$c\dot{x}(t) + \tilde{x}(t) = 0 \tag{6.22}$$

Comparing the solutions to (6.21) and (6.22) at the same initial conditions, i.e. $x(t_1) = \tilde{x}(t_1)$, establishes for all $t \geqslant t_1$ the validity of the inequality $|x(t) - \tilde{x}(t)| \leqslant \eta = 1/\alpha$, which leads us to the desired conclusion.

With α at infinity, beginning from a certain time instant the variation of error $x(t)$ in the considered S_μ system no longer differs from the variation of error $\tilde{x}(t)$ in the ideal system, so that the bound (6.3) holds. The system proper, however, now goes over to the class of S_μ systems with bang-bang COFB law since at $\alpha = \infty$ the expression for $\mu(t)$ takes the form

$$\mu = -\mu_0 \operatorname{sgn}(\sigma x) \tag{6.23}$$

The block diagram picturing such ultimate gain S_μ systems is represented in Fig. 6.6.

A qualitative treatment of S_μ systems with bang-bang COFB will be given in one of the following sections. For the time being we confine ourselves to noting only that such control systems relate to the class of automatic control systems with variable structure. When the relationships (6.18) are satisfied, at each point of the line $\sigma(x) = 0$ such systems are known [24] to exhibit a sliding mode described by the equation of motion (6.22). This fact reveals still

another difference in the role of constraints on the control signal in the CFB and COFB loops. In the first case, as the gain grows to infinity (see Chapter 5) such a constraint brings about a sliding mode appearing on a limited line section so that the controlled processes become independent of the $\mathcal{P}_x$ parameters as soon as they enter this section. In the second case, the entire line $\sigma(x) = 0$ is the line of sliding and the independence of the controlled processes of the $\mathcal{P}_x$ parameters occurs as soon as they enter this line. It is worth noting here that as the result of the limit transition as $\alpha \to \infty$ the control $u(t)$ becomes discontinuous, and, consequently, the condition $u(t) \in \mathcal{F}_c$ is violated.

In conclusion, the constraint on control introduced in the COFB loop favorably influences the behavior of the control system with time-invariant proportional coordinate-operator feedback, simplifies analysis of such control systems, and allows for the constraints on gains in the coordinate feedback loop. For the class of continuous control signals $u(t)$, the system can be made only $\mathcal{A}_\mu$ quasicontrolled. As for the properties of the system being $\mathcal{A}_\mu$ controlled and, to say more, exponentially $\mathcal{A}_\mu$ controlled, these in general are unattainable in this case.

With time invariant COFB laws and $u(t) \in \mathcal{F}_c$, the system can be made $\mathcal{A}_\mu$ controlled with finite values of design parameters. These laws will be covered in the ensuing section.

6.3 Nonlinear Time-Invariant Coordinate-Operator Feedback

Let us formulate an $\mathcal{A}_\mu$-algorithm of control with time-invariant coordinate-operator feedback (COFB) such that would generate a continuous control $u(t)$ and ensure the system being exponentially controlled. As follows from the previous consideration, to achieve this goal we need at least to ensure for the respective S_μ system the unique equilibrium position at the origin O_x. This, of course, may be achieved in various ways. From what we already know it seems appropriate to adopt the approach based on the control law (6.9) but not over the entire phase plane $(x, \dot{x})$, rather over that portion containing O_x and having no other equilibrium positions possible for such a nonlinear dynamic system. The rest of the phase plane should be covered by COFB laws such that denude the control system of any equilibrium points. It will be in order to note that unless stated otherwise, the CFB law is taken in the form $u = \bar{k}\mu x$ and, consequently, the control algorithm is defined by the COFB law.

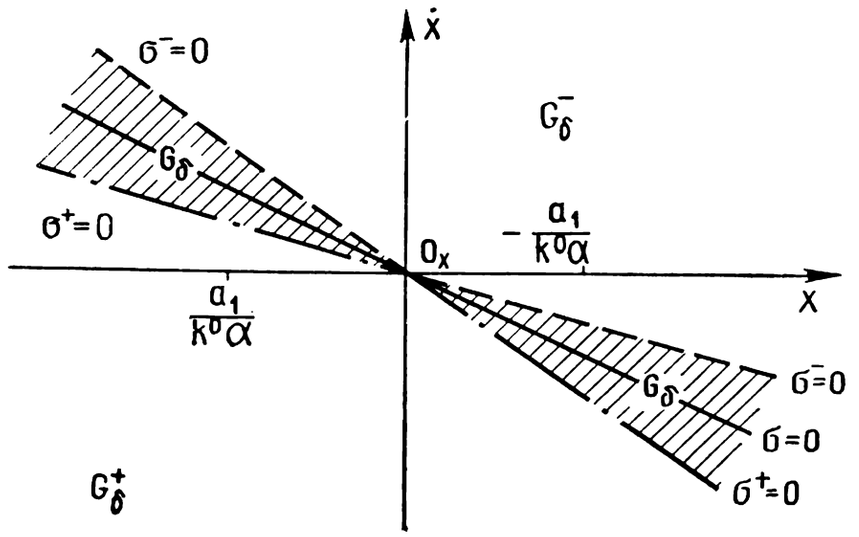

Figure 6.7

When in addition the conditions are provided under which the controlled processes always reach that portion of the phase plane where the COFB law (6.9) is valid—this portion of the phase plane is invariant and for all processes belonging to it $\lim_{t\to\infty} \| x(t) \| = 0$ — then one may count upon an asymptotic solution of the control problem. Consider the possibilities offered by this approach.

On the phase plane we define a domain G_δ as follows:

$$G_\delta = x:\ \sigma^+(x)\sigma^-(x) \leqslant 0$$
$$\sigma^+(x) = \sigma(x) - \delta|x|, \quad \delta = \text{constant} \tag{6.24}$$
$$\sigma^-(x) = \sigma(x) + \delta|x|, \quad 0 \leqslant \delta < 1$$

Here as before $\sigma(x)$ is defined by the equation $\sigma(x) = -(c\dot{x} + x)$. For all $x \in G_\delta$ this function satisfies $|\sigma(x)| \leqslant \delta |x|$. The boundary of G_δ is specified by the equations $\sigma^+(x) = 0$ and $\sigma^-(x) = 0$ separating the domain from the areas $G_\delta^- = \{x:\ \sigma^-(x) < 0\}$ and $G_\delta^+ = \{x:\ \sigma^+(x) > 0\}$. It will be readily seen that for any $\delta \in [0, 1)$, $\sigma^-(x)$ is negative on the semiaxis $\{x:\ x > 0,\ \dot{x} = 0\}$, while the function $\sigma^+(x)$ is positive on the semiaxis $\{x:\ x < 0,\ \dot{x} = 0\}$, accordingly, the domain G_δ is located on the phase plane as illustrated in Fig. 6.7.

Inspection of Fig. 6.7 indicates that the singular points $|x^*_{2,3}| = -a_1/k^0\alpha$ that occur in the closed-loop control system with the law (6.9) all lie beyond the domain G_δ. Therefore, the linear COFB (6.9) will be fairly legitimate on the domain G_δ.

At the outset let us be clear as to the invariance of the domain G_δ for the considered COFB law operable for $x \in G_\delta$. Because in this case $\mu = -\alpha\sigma \operatorname{sgn} x$, the control u is given by the expression

$u = -\bar{k}\alpha\sigma \mid x \mid$ and for $x(t) \in G_\delta$ the behavior of the controlled processes is defined by the equation

$$\ddot{x} + a_2\dot{x} + a_1 x = \bar{k}\alpha\sigma \mid x \mid$$

It is quite obvious that G_δ will be invariant if the controlled processes taking off from the boundary of this domain do not leave this domain. In order to examine this topic analytically we invoke in our consideration for $x \in G_\delta$ and $\| x \| \neq 0$ a function $\lambda(x)$ defined as $\lambda(x) = \sigma(x)/\delta \mid x \mid$.

Since for $x \in G_\delta$ there holds $\mid \sigma(x) \mid \leqslant \delta \mid x \mid$, the function $\lambda(x)$ is of bounded variation, i.e. $\mid \lambda(x) \mid \leqslant 1$ for $x \in G_\delta$, and takes on its extreme values on the G_δ boundary, namely, $\lambda(x) = -1$ for $\sigma^-(x) = 0$ and $\lambda(x) = 1$ for $\sigma^+(x) = 0$. For each controlled process $x(t) \in G_\delta$ this function is a continuous time function $\lambda(t) = \lambda[x(t)]$ and the domain G_δ will obviously be invariant provided $\dot{\lambda}(t) \geqslant 0$ at $\lambda(t) = -1$ and $\dot{\lambda}(t) \leqslant 0$ at $\lambda(t) = 1$.

To check whether or not these conditions are met we derive the expression for $\dot{\lambda}(t)$. From the definition of $\lambda(x)$

$$\dot{\lambda}(t) = \frac{\dot{\sigma}(t) - \delta\lambda(t)\operatorname{sgn} x(t)\,\dot{x}(t)}{\delta \mid x(t) \mid}$$

Substituting the expression for $\dot{\sigma}(t)$, $\dot{x} = -(\sigma + x)/c$ for $\dot{x}(t)$, and collecting like terms yields

$$\dot{\lambda} = -\left(c\bar{k}\alpha \mid x \mid + \frac{ca_2 - 1 - \xi}{c}\right)\lambda + \frac{1}{\delta}\left(ca_1 - \frac{ca_2 - 1 - \xi}{c}\right)\operatorname{sgn} x$$

where, for convenience, $\xi = \delta\lambda \operatorname{sgn} x$. Observing that $\delta < 1$, $c > 0$, and $a_1 < 0$, the derived expression readily reveals that for sufficiently small $\mid x \mid$ the sign of the derivative $\dot{\lambda}(t)$ on the G_δ boundary where $\mid \lambda \mid = 1$ is determined by the sign of the second term on the right-hand side, which can be arbitrary even subject to the condition $\inf ca_2(t) > 1 + \delta$ for $t \geqslant t_0$. Hence, in some neighborhood of O_x the controlled processes $x(t)$ tend to leave the domain G_δ which is not invariant for the COFB law at hand. We note that this result is valid for other configurations of the G_δ type of domain.

Invariance of G_δ may be provided by creating sliding modes on the sections of the G_δ boundary where $\dot{\lambda}\lambda > 0$. Specifically, G_δ will be invariant with the following COFB law:

$$\begin{aligned} \mu &= -\alpha\sigma \operatorname{sgn} x \quad \text{for} \quad x \in G_\delta \\ &= -\mu_0 \operatorname{sgn}(\sigma x) \quad \text{for} \quad x \notin G_\delta \end{aligned} \tag{6.25}$$

where μ_0 is a sufficiently large positive number

The COFB law (6.25) is, however, discontinuous and the control function $u(\mu, x)$ also suffers discontinuities (giving rise to the mentioned sliding mode on some sections of the G_δ boundary). Consequently, the control $u(t)$ will also be a discontinuous time function. This is inconsistent with the earlier requirement of $-u(t) \in \mathcal{F}_c$, where $\mathcal{F}_c$ is the class of functions continuous over $[t_0, \infty)$.

The performed examination, therefore, provides a supportive evidence to our assertion that it is impossible to ensure the asymptotic solution to the control problem with a continuous control action $u(t)$ in the class of linear time-invariant COFB laws. It would be interesting, therefore, to look at the possibilities of *nonlinear* static and dynamic COFB in handling this problem.

The evaluation of control systems with dynamic COFB laws is reserved for the next chapter, for the moment we shall focus on process behavior determined by the nonlinear time-invariant COFB law of the form

$$\mu = -\mu_0 \operatorname{sat}(\alpha\sigma/|x|) \operatorname{sgn} x, \quad \alpha\delta = 1 \tag{6.26}$$

where sat $(\cdot)$ is the saturation function defined in (6.12).

This COFB law is advantageous in that the control function $u(\mu, x)$ is a continuous function of position of state point x, and consequently, the control action $u(t)$ will also be a continuous function of time. In addition, the operator variable μ is bounded for (6.26), hence the CFB loop gain is also bounded—in agreement with the definition of the complete solution of the control problem. Finally, the variable μ takes on constant values of different sign in the areas $G_\delta^{\pm}$ and is variable only for those points x of the phase plane which belong to G_δ. With the COFB law (6.26) the control function has the form

$$u = -\bar{k}\mu \,|\, x \,|\, \operatorname{sat}(\alpha\sigma/|x|)$$

therefore the differential equations that govern the behavior of controlled processes $x(t)$ in the S_μ system may be represented as follows:

$$\begin{aligned} \ddot{x} + a_2\dot{x} + a_1 x &= \bar{k}\mu_0 |x| && \text{for } x \in G_\delta^+ \\ &= \bar{k}\mu_0 \alpha\sigma(x) && \text{for } x \in G_\delta \\ &= -\bar{k}\mu_0 |x| && \text{for } x \in G_\delta^-, \quad t \geqslant t_0 \end{aligned} \tag{6.27}$$

Let us analyze qualitatively how controlled processes perform in such an S_μ system. Consider first the case of $x(t_0) \in G_\delta^-$ (the case of $x(t_0) \in G_\delta^+$ can be analyzed in a similar manner). First of all we note that the last equation in (6.27) is consistent with the situation

at hand and coincides with the last equation in (6.15). Consequently, the variation of $\sigma(t) = \sigma(x(t))$ along a controlled process under the stated initial conditions is defined by equation (6.19). From this equation it readily follows that given (6.18) are satisfied $\dot{\sigma}(t)$ is positive if $x(t) \neq O_x$. Consequently, the function $\sigma(t)$ monotonously increases on all controlled processes from G_δ^- and there may be two types of behavior for such processes. One of them, later referred to as $x^{\mathrm{I}}(t)$ processes, takes a finite time to reach the boundary $\sigma^-(x) = 0$ of the domain G_δ; for the other processes, denoted as $x^{\mathrm{II}}(t)$ processes, $\lim_{t \to \infty} \sigma(x^{\mathrm{II}}(t)) = 0$. Geometrically, these facts immediately follow from examination of Fig. 6.7.

From the condition $\lim_{t \to \infty} \sigma(x^{\mathrm{II}}(t)) = 0$ it follows that $\lim_{t \to \infty} \| x^{\mathrm{II}}(t) \| = 0$, i.e. these processes converge to O_x. Indeed, the first component $x(t)$ of $x^{\mathrm{II}}(t)$ and the function $\sigma(t)$ are connected by a differential relation as has been demonstrated by (6.21), namely $c\dot{x}(t) + x(t) = -\sigma(t)$; here we have dropped the superscript for convenience. This equation indicates that for any $x(t_0)$, $\lim_{t \to \infty} x(t) = 0$ provided $\sigma(t) \to 0$ as $t \to \infty$, and, consequently, $\lim_{t \to \infty} \dot{x}(t) = 0$ which is what we have sought to establish.

Let us now look at the behavior of $x^{\mathrm{I}}(t)$ processes beginning from the time instant when they first arrive at the boundary of G_δ from the area G_δ^-. Let t_1 denote this time instant. For analysis we need the expression of $\dot{\sigma}^-(x^{\mathrm{II}}(t))$ on the boundary of G_δ, i.e., where $\sigma^-(x) = 0$. Recalling the expression for $\sigma^-(x)$, we get

$$\dot{\sigma}^-(x(t)) = \dot{\sigma}(x(t)) + \delta |x(t)|^{\cdot} = \dot{\sigma}(x(t)) + \delta \operatorname{sgn} x(t)\, \dot{x}(t)$$

Here and below in similar situations we assume $|x(t)|^{\cdot} = d|x(t)|/dt = \operatorname{sgn} x(t)\, \dot{x}(t)$.

Because in the situation concerned $x \in G_\delta$, then $\sigma = -\delta |x|$ and motion in the S_μ system for $t \geqslant t_1$ is governed by the second equation in (6.27). Using this equation to compute the derivatives in the expression for $\dot{\sigma}^-(t)$ with the substitution $\dot{x} = -(\sigma + x)/c$ we finally arrive at the following expression computed at $\sigma^- = 0$:

$$\dot{\sigma}^-|_{\sigma^-=0} = \delta \left(c\bar{k}\mu_0\alpha + \frac{ca_2 - 1 + \delta\xi}{c} \right) |x| + \left(ca_1 - \frac{ca_2 - 1 + \delta\xi}{c} \right) x, \quad t \geqslant t_0 \tag{6.28}$$

where $\xi = \operatorname{sgn} x$.

Observing $\alpha\delta = 1$ in (6.28) and the following inequalities (henceforth assumed to be satisfied and this fact will not be notified specifically)

$$c\bar{k}\,\mu_0 > \max_{|\xi|=1} \sup_{t\geqslant t_0} \left[ca_1(t) - \frac{ca_2(t) - 1 + \delta\xi}{c} \right] \tag{6.29a}$$

$$\inf_{t\geqslant t_0} ca_2(t) > 1 + \delta \tag{6.29b}$$

we establish that $\dot{\sigma}^-(t)$ is positive on the boundary $\sigma^-(x) = 0$. This implies that the controlled processes in the S_μ system under study cannot enter the area G_δ^- from the boundary of G_δ. In a similar manner it can be demonstrated that the controlled processes do not enter the area G_δ^+ from the boundary $\sigma^+(x) = 0$.

To summarize, we have established that when the conditions (6.29) are met, in the S_μ system under consideration the controlled processes $x(t)$ once they have reached the domain G_δ at some time instant never leave this domain, i.e., $x(t) \in G_\delta$ for all subsequent t. In other words, G_δ is an *invariant* domain for such S_μ system.

For the x^1 processes, the analysis of their further behavior, i.e. for $t \geqslant t_1$, is carried out with the use of the differential constraint (6.21) and the inequality $|\sigma(t)| \leqslant \delta |x(t)|$ valid as long as $x(t) \in G_\delta$. That is, our tools in the evaluation of the asymptotic behavior of these processes will be

$$c\dot{x}(t) + x(t) = -\sigma(t), \quad |\sigma(t)| \leqslant \delta |x(t)|, \quad t \geqslant t_1 \tag{6.30}$$

The general solution to (6.30) has the form

$$x(t) = x(t_1) \exp\left(-\frac{t-t_1}{c}\right) - \frac{1}{c}\int_{t_1}^{t} \exp\left(-\frac{t-\tau}{c}\right)\sigma(\tau)\,d\tau$$

From this equation we go over to the following inequality valid for $t \geqslant t_1$:

$$|x(t)| \leqslant |x(t_1)| \exp\left(-\frac{t-t_1}{c}\right) + \frac{\delta}{c}\int_{t_1}^{t} \exp\left(-\frac{t-\tau}{c}\right)|x(\tau)|\,d\tau$$

Multiplying both sides of this inequality by $\exp(t/c)$, applying the Gronwall-Bellman lemma [96], and multiplying the result by $\exp(-t/c)$ yields the final estimate for the solution (6.30)

$$|x(t)| \leqslant |x(t_1)| \exp\left[-\frac{1-\delta}{c}(t-t_1)\right], \quad t \geqslant t_1 \tag{6.31}$$

Now we recall that $|\dot{x}| \leqslant (1+\delta)|x|/c$ for $x \in G_\delta$ and use (6.31) to arrive at the exponential estimate which holds for the considered process $x(t)$ for $t \geqslant t_1$, viz.,

$$\|x(t)\| \leqslant \frac{1+c+\delta}{c} \|x(t_1)\| \exp\left[-\frac{1-\delta}{c}(t-t_1)\right] \tag{6.32}$$

If now $0 < \delta < 1$, then (6.31) immediately yields

$$\lim_{t \to \infty} \| x(t) \| = 0$$

This equality concludes the proof of convergence of all controlled processes in the system on hand to the origin of the phase space $\mathbb{R}^2_x$.

Let us demonstrate that the origin O_x is asymptotically stable in the large. To achieve this goal we need to substantiate the validity of the bound (4.4a) for every controlled process $x(t)$ of the S_μ system under study. Consider an arbitrary controlled process which at $t = t_0$ leaves an arbitrary sphere $\| x(t_0) \| \leqslant \gamma$, γ being a positive constant. If $x(t_0) \in G_\delta$ then for all $t \geqslant t_0$ the inequality (6.32) is satisfied and, consequently,

$$\| x(t) \| \leqslant \frac{1+c+\delta}{c} \| x(t_0) \|$$

is valid. We suppose, therefore, that $x(t_0) \notin G_\delta$ moreover, for definiteness, let $x(t_0) \in G_\delta^-$ (for $x(t_0) \in G_\delta^+$, the results are similar). Then for $\dot{\sigma}(t)$ taken along such a controlled process, we may write (6.19) and provided (6.29) are satisfied, we obtain $\dot{\sigma}(t) > 0$ for $t \geqslant t_0$. Hence, $\sigma(t)$ increases and for all $t \geqslant t_0$ we have $\sigma(t) > \sigma(t_0)$.

Let $\boldsymbol{c}$ denote the row vector $[1, c]$, then $\sigma(x) = -\boldsymbol{c}x$. From the above inequality $\sigma(t) > \sigma(t_0)$ we have for $t \geqslant t_0$ the self-evident series

$$\boldsymbol{c}x(t) < \boldsymbol{c}x(t_0) \leqslant c_m \| x(t_0) \|$$

where $c_m = \max(1, c)$. On the other hand, there always holds $|\dot{x}| \leqslant (|x| + |\sigma|)/c$ and, consequently,

$$\| x \| \leqslant \frac{1+c}{c} |x| + \frac{1}{c} |\sigma|$$

Since in the case under investigation $\delta |x| < \boldsymbol{c}x$ and $|\sigma| = \boldsymbol{c}x$, the previous inequality leads us to

$$\| x(t) \| \leqslant \frac{1+c+\delta}{\delta c} \boldsymbol{c}x(t)$$

valid for all $t \geqslant t_0$. Incorporating this expression in the inequality $\boldsymbol{c}x(t) \leqslant c_m \| x(t_0) \|$ carries it to

$$\| x(t) \| \leqslant \frac{1+c+\delta}{c\delta} c_m \| x(t_0) \| \qquad (6.33)$$

valid for all $t \geqslant t_0$ for type-II controlled processes.

For type-I processes, i.e. those for which there exists a finite t_1 when $x(t_1) \in G_\delta$, the desired bound of type (4.4a) follows from (6.32) upon substitution of the right-hand side from (6.33) with

$= t_1$. This argument proves the fact of stability for the point O_x, and taken together with the convergence of controlled processes to this point established above, it proves its asymptotic stability in the large.

The performed investigation suggests that the exponential estimate (6.32) may be given under the specified conditions for type-I processes only, i.e. those for which beginning with a certain time instant $x(t) \in G_\delta$. In the circumstances, if δ is sufficiently small, say $\delta \ll 1$, then the dynamic behavior of such processes depends only insignificantly on the $\mathcal{P}_x$ parameters in the sense of the earlier given definition. Conversely, a similar conclusion on the type-II process behavior, i.e. those for which $x(t) \in G_\delta^{\pm}$, is, generally speaking, impossible altogether. Therefore the $\mathcal{A}_\mu$-algorithm treated in this section is incapable of providing the complete solution to the considered control problem, that is, fails to satisfy the entire set of requirements (6.1)-(6.3).

To summarize we established the property of $\mathcal{P}_x$-processes being steady $\mathcal{A}_\mu$ controlled by the $\mathcal{A}_\mu$ control algorithm under the specified conditions and constraints. This fact can be written in shorthand as follows:

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\mu}\{\mathcal{F}_c,\ I_2;\ N\} \tag{6.34}$$

where $N = (1 + c + \delta)^2\, c_m/c^2\delta$. Moreover for type-I processes there exists the exponential bound (6.32).

It is worth noting that the property (6.34) is valid for any character of variation of the $\mathcal{P}_x$ parameters $a_1(t)$ and $a_2(t)$ within the ranges (4.2). For any values of a_1 and a_2 fixed within (4.2), a more detailed study may be carried out to establish for x^{I} processes the existence, for a positive constant c_k, of an asymptote $c_k\dot{x} + x = 0$ such that $\{x:\ \boldsymbol{c}_k\boldsymbol{x} = 0\} \subset G_\delta$, $\boldsymbol{c}_k = [1,\ c_k]$. (This conclusion follows from the similarity of the equations for the S_μ system and for the system with the $\mathcal{A}_k$ algorithm investigated in the previous chapter.) The estimate (6.32) could be refined for any such $\mathcal{P}_x$-process, but because for fixed a_1 and a_2 of (4.2) the asymptote $\boldsymbol{c}_k\boldsymbol{x} = 0$ can take on arbitrary position in the domain G_δ, there is no way of improving this bound for the *entire family* of $\mathcal{P}_x$-processes at finite α and $\bar{k}$.

Let us look now at how the properties of the S_μ system concerned change with increasing gain α or $\bar{k}$. Let at first $\bar{k} \to \infty$, then in the limit, subject to a constraint imposed on the control in the CFB loop, the S_μ system becomes a bang-bang control system, whose behavior has been covered in Section 5.3, and, consequently, the objective set in (6.3b) is not achievable.

Assume then that the gain $\bar{k}$ is limited so that $\bar{k}\mu_0 \leqslant k^0$, and the gain $\alpha \to \infty$. It is not hard to see that in this case the COFB law

(6.26) takes the form of (6.23) and the ultimate gain S_μ system has the functional block diagram shown in Fig. 6.6. This is an S_μ system with a bang-bang COFB law, for which, as has been revealed in the previous section, the property (6.3b) is also unattainable as the control action for this system is a discontinuous time function, i.e. $u(t) \notin \mathcal{F}$. At sufficiently large, but finite α values, the behavior of this system can be brought nearer, however close, to that indicated in (6.3b); recall that $\delta = 1/\alpha$. It should be emphasized that this fact takes place only with a constraint imposed on the control in the COFB loop. When this constraint is lifted, the considered $\mathcal{A}_\mu$ algorithm becomes an $\mathcal{A}_k$ algorithm—the behavior of such control systems has been discussed in Chapter 5.

Thus, the constraint on control, as introduced in the COFB loop, offers an arbitrary accurate approach to the behavior required in (6.3b), though without achieving this absolutely. It will be recalled that a similar constraint in the CFB loop prevents the system from achieving a similar behavior.

To conclude this topic we wish to briefly touch upon the effect of the dynamic nonlinearities, mentioned in Sections 5.6-5.7, on the behavior of the S_μ systems under study. If the controlled process is such that a linear zone of COFB is operable, then its behavior is governed by a linear differential equation outwardly coinciding with the equations of the respective systems with $\mathcal{A}_k$ algorithms covered in Sections 5.6-5.7. For fixed $\mathcal{P}_x$ parameters a_1 and a_2, this implies that this case is amenable to analysis with the characteristic equations for closed-loop control systems, outlined in Sections 5.6-5.7, subject to substitution $k = \overline{k}\mu_0 \leqslant k^9\mu_0$.

Whenever the stability conditions either (5.12) for the first type of nonlinearity or (6.2) for the second type of nonlinearity are unattainable, the projection of the controlled process onto the phase plane—dynamic nonlinearities increase the dimensionality of the phase space—leaves the domain G_δ, i.e. this domain is no longer invariant. Therefore, when the designer is faced with such nonlinearities, the question of whether or not the control problem is solvable, and in what sense, remains open. These remarks conclude our investigation in this section. In what follows we shall concern ourselves with the behavior of S_μ systems with bang-bang coordinate-operator feedback.

6.4. Bang-Bang Coordinate-Operator Feedback

The so far evaluated S_μ systems with limited control in the COFB loop degenerate into S_μ systems with bang-bang COFB when the gain in this COFB loop tends to infinity. Let us consider these S_μ systems in depth. To specify an S_μ system with bang-bang

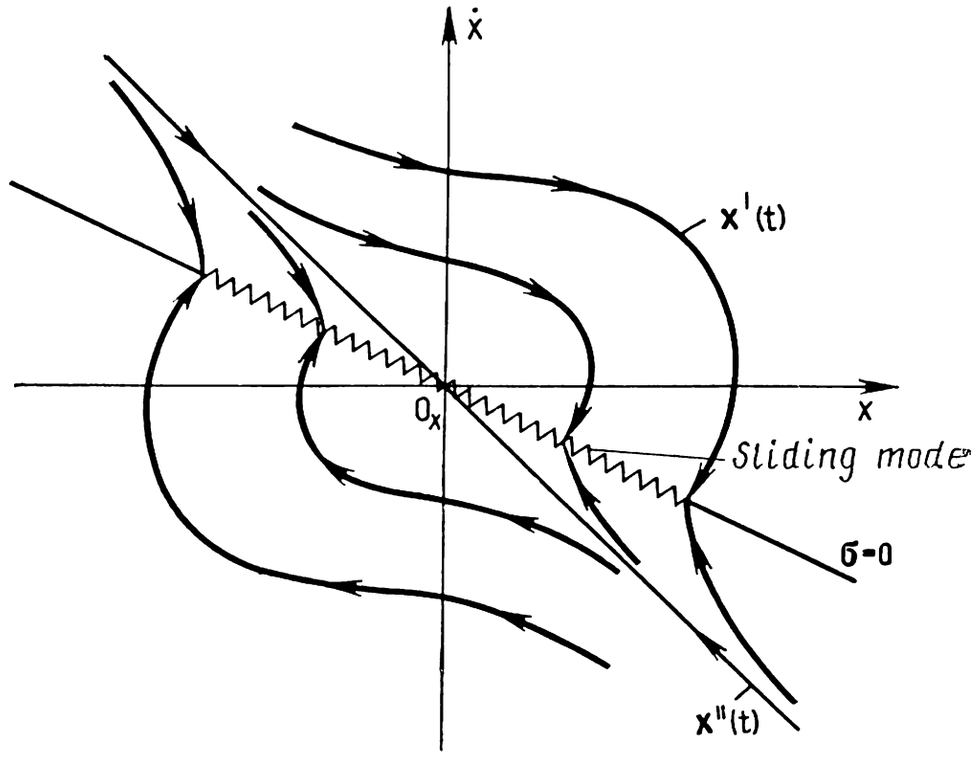

Figure 6.8

COFB we are to define the equation of the $\mathcal{P}_x$-process, expressions (6.5), (6.6) and (6.23), the latter defining a bang-bang COFB law. The block diagram for such a system is presented in Fig. 6.6. The controlled processes x (t) are determined as the solutions to the differential equation

$$\ddot{x} + a_2\ (t)\ \dot{x} + a_1\ (t)\ x = \overline{k}\mu_0 \mid x \mid \operatorname{sgn} \sigma, \quad t \geqslant t_0 \tag{6.35}$$

where $\sigma = -(c\dot{x} + x)$.

This is a differential equation with a discontinuous right-hand side. Its solution is sought in conformity with the definition given in Ref. 93. This reference also outlines the conditions of solvability for (6.35) and the continuation of the solution onto the semiaxis $t \geqslant t_0$.

In terms of automatic control theory, equation (6.35) describes a system of variable structure. Such systems are well investigated and documented (see, e.g. Refs. 24, 25, 96, 97 and references cited therein). Henceforth we will refer to any system with variable structure as a VS system. Thus any S_μ system with bang-bang COFB is a VS system.

For any fixed values of $\mathcal{P}_x$ parameters from within (4.2) and (6.18) being satisfied, the qualitative phase portrait of the S_μ system on hand is known [24] to be of the form indicated in Fig. 6.8. It will be seen that two types of controlled processes x (t) are possible in the S_μ system with bang-bang COFB.

The first type of processes, $x^{\mathrm{I}}(t)$, having started from any initial conditions, in a finite time reach the surface of discontinuity of the right-hand side of (6.35), $G_0 = \{x\colon \sigma(x) = -cx = 0\}$; where a sliding mode occurs to remain at all subsequent t.

For each x^{I} process, beginning from t_1 when a sliding mode occurs, $cx^{\mathrm{I}}(t) \equiv 0$, the vector functions $x^{\mathrm{I}}(t)$ being absolutely continuous, and in agreement with Ref. 93 their time variation law can be determined by solving

$$\dot{cx}(t) + x(t) = 0, \quad t \geqslant t_1 \tag{6.36}$$

Therefore, any x^{I} process for all $t \geqslant t_1$ may be bounded as

$$\| x^{\mathrm{I}}(t) \| \leqslant \frac{1+c}{c} \| x^{\mathrm{I}}(t_1) \| \exp - \frac{i - i_1}{c} \tag{6.37}$$

From (6.36) it will be evident that the dynamic behavior of such controlled processes are independent of the $\mathcal{P}_x$ parameters beginning only from time instant t_1, individual for each $x^{\mathrm{I}}(t)$ process.

The type-II control processes, $x^{\mathrm{II}}(t)$, asymptotically approach the origin O_x. A formal substantiation of this property of x^{II} processes may be carried out with account of (6.18) following the line of the argument used in the preceding section for asymptotic behavior analysis of type-II processes.

Thus in the considered S_μ system, for any parameters $a_1(t)$ and $a_2(t)$ from (4.2), the controlled processes converge to O_x. To prove that this point is stable, one may invoke the line of argument from the previous section. It will be sufficient to note that the inequalities (6.18) allow the existence of a positive number $0 < \delta_1 < 1$ such that for the S_μ system on hand the domain $G_{\delta 1} = \{x\colon |\sigma(x)| \leqslant \delta_1 |x|\}$ is invariant. Therefore, similar to the argument in the previous section, it will not be hard to establish that (4.4a) is valid for all processes in such system as long as $N = N_1 = (1 + c + \delta_1)^2 c_{\mathrm{m}}/\delta_1 c^2$. Hence the point O_x is stable.

It is worth noting that equation (6.36) and hence the independence of x^{I} dynamics of plant parameters are valid only when a sliding mode occurs, and the control function $u(t)$ is discontinuous at each point, i.e. $u(t) \in \mathcal{F}_s$. Therefore, for x^{I} processes, the above property may be stated in shorthand as follows:

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\mu} \{\mathcal{F}_s,\ I_2;\ (1+c)/c,\ 1/c\}_{G_0} \tag{6.38}$$

Recognizing also that the dynamic behavior of x^{II} processes depends substantially on the $\mathcal{P}_x$ parameters, we ought to confess that no complete solution of the control problem can be achieved in the class of S_μ systems with bang-bang COFB, i.e. in the class of considered VS systems.

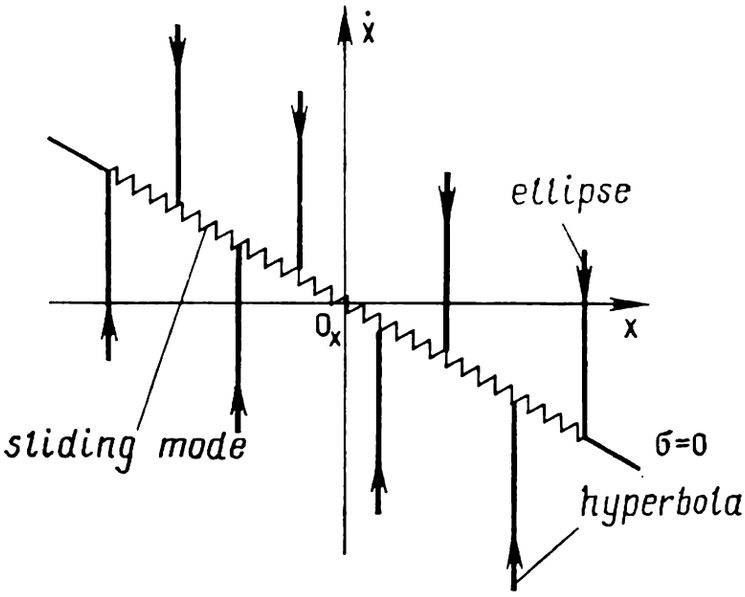

Figure 6.9

For all similarity of the phase portraits in Figs. 5.3 and 6.7, there is a principal difference between them. In Fig. 5.3 the phase trajectories are continuous and serve as asymptotes for the curves of parabolic equations. The phase trajectories in Fig. 6.7 are also continuous, but they are asymptotic for the curves of hyperbolic type for $\sigma x > 0$ and for the elliptic curves for $\sigma x < 0$. This difference becomes of decisive importance in account of nonlinearities and constraints.

On the whole, the property (6.38) does not hold for all processes of such system, so that this time

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\mu}\{\mathcal{F}_s,\ I_2;\ N_1\} \tag{6.39}$$

The statement (6.38) may be insured on the whole also from the instant $t = t_0$ in the absence of gain limitations in the CFB and COFB loops and a constraint on the control in the CFB loop. Indeed, let μ_0 grow without limit, then with μ_0 at infinity the phase portrait will be as shown in Fig. 6.9 rather than that in Fig. 6.8. For such an ultimate gain S_μ system, each controlled process immediately appears in the domain G_0 and after this the estimate (6.37) is valid for all $t \geqslant t_0$. Of course, this possibility cannot be realized in practice, being prevented by various nonlinearities and constraints, say of the collection delineated in the previous chapter. Specifically, with a constraint on the control in the CFB loop, the ultimate gain S_μ system will be a bang-bang system with all the salient features that have been discussed in Section 6.3.

6.5 Bang-Bang Coordinate-Operator Feedback with Constraints and Dynamic Nonlinearities

In Section 6.4 we have evaluated the influence of a constant time lag (dead time) on the behavior of a closed-loop control system with a bang-bang CFB law (termed the $\mathcal{A}_\tau$ control algorithm) and found out that for systems with such control algorithms it is only a Δ-accurate solution to the control problem that can be ensured. More accurately, the properties of the respective control algorithms are described by the succinct expression $\mathcal{P}_x \in \mathcal{C}_{(\Delta, \mathcal{A}_\tau)} \{\mathcal{F}_{\text{pc}}, I_2; N\}$. This section will be focused on the features introduced into the operation of the S_μ system with bang-bang COFB (Fig. 6.6) by a constant time lag and will compare the behavior of controlled processes in such S_μ system and in that with the $\mathcal{A}_\tau$ control algorithm.

In the presence of switching lag, i.e. dead time, the controlled processes $x(t)$ in S_μ systems with bang-bang COFB are defined as the solutions of

$$\ddot{x}(t) + a_2(t)\dot{x}(t) + a_1(t)x(t) = \bar{k}\mu_0 \mid x(t) \mid \operatorname{sgn} \sigma_\tau(t) \qquad (6.40)$$

for $t \geqslant t_0$ where the function $\sigma_\tau(t)$ is defined as $\sigma_\tau(t) = \sigma(t - \tau)$, $\sigma(t) = -[x(t) + c\dot{x}(t)]$, $\tau = \text{constant} > 0$, $\sigma(t) = \varphi(t)$ for $t \in \in [t_0 - \tau, t_0]$, and $\varphi(t)$ is the initial function. For simplicity we assume that, for $t \in [t_0 - \tau, t_0]$, $\varphi(t) \in \mathcal{F}_c$ and $\operatorname{sgn} \varphi(t) = \operatorname{sgn} \sigma(t_0) = -\operatorname{sgn} [c\dot{x}(t_0)]$, and $x(t_0) = [x(t_0), \dot{x}(t_0)]^T$ is the initial vector.

Equation (6.40) and the range constraints (4.2) for the parameters $a_1(t)$ and $a_2(t)$ specify the system of $\mathcal{P}_x$-process control using an $\mathcal{A}_\mu$ control algorithm with two-step COFB and operating with a constant dead time in two-position element switching. Like with the control system evaluated in Section 5.4, the phase space of the system subject to the dynamic equation (6.40) is of infinite dimensionality. Therefore, to retain the possibility of using the $(x, \dot{x})$ plane for graphical representation of motions in this control system we confine ourselves to examining only part of solutions to (6.40). More specifically, like in Section 5.4, we shall focus on the solutions for which the zeros of $\sigma(t)$ are spaced in time by more than the dead time τ. It was this argument that dictated earlier the choice of the sign for the initial function $\varphi(t)$.

The behavior of the selected solutions to differential equations such as (6.40) has been studied for variable structure systems, see e.g. [24]. The general findings of VS system theory are as follows. No sliding mode occurs on the domain G_0, and the closed-loop con-

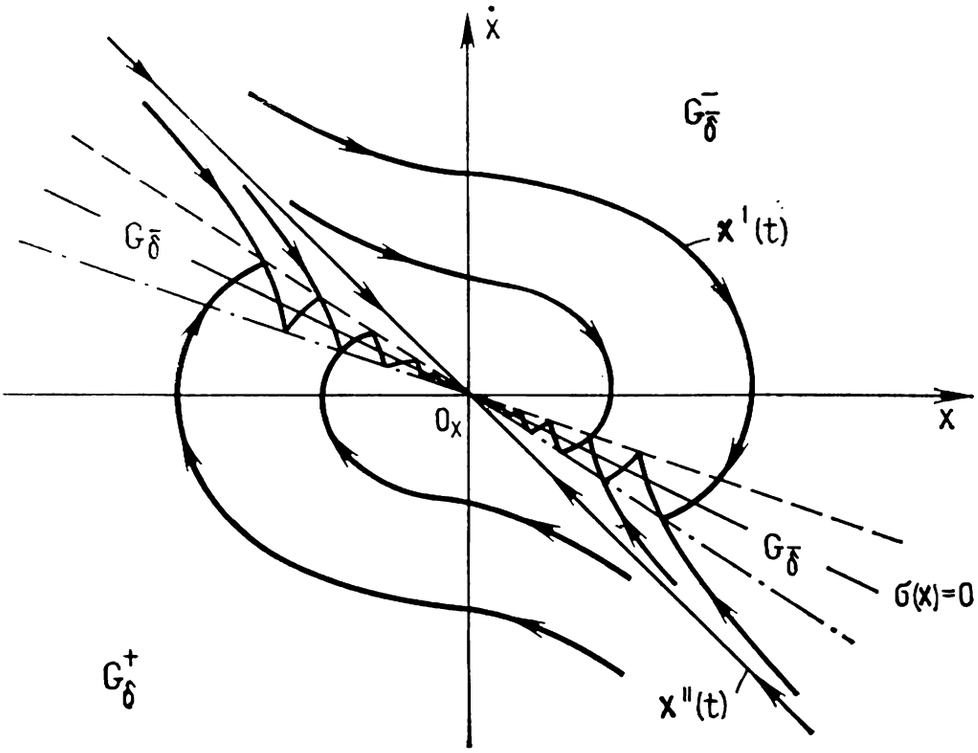

Figure 6.10

trol system exhibits a switching mode. For some relations between control system parameters (i.e., the parameters of $\mathcal{P}_x$-process and the design parameters or control algorithm parameters) and a sufficiently small value of τ, the so-called *real sliding mode* can occur in the neighborhood of G_0. Then all the considered controlled processes asymptotically approach zero so that the control problem is solvable asymptotically.

A qualitative insight into the behavior of solutions to (6.40), for any fixed parameters a_1 and a_2 from (4.2), can be gleaned from nspection of Fig. 6.10, where $G_{\bar{\delta}}$ represents the domain of the rea sliding mode—it is confined between the dashed and chained lines. This domain has a structure coinciding with that of G_δ, being defined by

$$G_{\bar{\delta}} = \{x: \mid \sigma(x) \mid \leqslant \delta \mid x \mid\} \tag{6.41}$$

where $\bar{\delta} \geqslant 0$ is a constant depending on τ and being 0 when $\tau = 0$.

Let us examine this S_μ system with the purpose to establish the relationship between δ and τ. To be more specific, we need to establish relationships between the parameters of S_μ system defined by (6.40) with the parameters $a_1(t)$ and $a_2(t)$ from (4.2) and a constraint on the dead time τ such that for any controlled process $x(t)$ ensure from some time instant $t \geqslant t_0$ that $x(t) \in G_{\bar{\delta}}$ at some constant $\bar{\delta} > 0$, or else $\lim\limits_{t\to\infty} \| x(t) \| = 0$.

It will be noted that controlled processes for which the last of the above properties holds are processes of second type, $\boldsymbol{x}^{\mathrm{II}}(t)$, in the sense of the definition given in the previous section. They will be examined with the argument similar to that exploited in Section 5.4. Accordingly, we shall limit ourselves mainly to analyzing the control processes of first type, $\boldsymbol{x}^{\mathrm{I}}(t)$, the respective superscripts, therefore, being dropped for simplicity.

Consider a controlled process $\boldsymbol{x}(t)$ for $t \geqslant t_0$ and define the time function

$$\lambda(t) = \lambda[\boldsymbol{x}(t)] = \sigma[\boldsymbol{x}(t)]/|x(t)| \tag{6.42}$$

Geometrically, $\lambda(\boldsymbol{x})$ may be treated as follows. For every point $\boldsymbol{x} \in G_\delta$ the number $\lambda(\boldsymbol{x})$ indicates the slope of the line passing through the origin O_x and the point $\boldsymbol{x}$.

When $\boldsymbol{x}(t) \in G_\delta$ on a certain time interval $[t_1, t_2]$ and $\|x(t)\| \neq 0$, the function $\lambda(t)$ is defined and continuous on this time interval, and there exists the estimate $|\lambda(t)| \leqslant \bar{\delta}$.

Because for the S_μ system being considered $u(t) \in \mathcal{F}_{\mathrm{pc}}(t)$ for all $t \geqslant t_0$ (recall that $\mathcal{F}_{\mathrm{pc}}$ is the class of piecewise continuous functions defined for all $t \geqslant t_0$), the quantity $\dot{\lambda}(t)$ has sense for all t in $[t_1, t_2]$. Let us derive the expression for this derivative. From (6.42) we have for $t \in [t_1, t_2]$

$$\dot{\lambda}(t) = \frac{\dot{\sigma}[\boldsymbol{x}(t)]}{|x(t)|} - \lambda(t)\frac{|x(t)|^{\cdot}}{|x(t)|} \tag{6.43}$$

Computing the derivatives on the right-hand side and substituting for $\dot{x}(t)$ its expression $\dot{x}(t) = -[1 + \lambda(t)\operatorname{sgn} x(t)]\, x(t)/c$, which follows from (6.42) and the expression for $\sigma(\boldsymbol{x})$, we get the following equation for $\dot{\lambda}(t)$ valid for all t from $[t_1, t_2]$ (the argument t is dropped here and below):

$$\begin{aligned}\dot{\lambda} = {} & ca_1 \operatorname{sgn} x - a_2(1 + \lambda \operatorname{sgn} x)\operatorname{sgn} x \\ & + \frac{(1 + \lambda \operatorname{sgn} x)^2}{c}\operatorname{sgn} x - c\bar{k}\mu_0 \operatorname{sgn} \sigma_\tau\end{aligned} \tag{6.44}$$

Analysis of the right-hand side of (6.44) readily indicates that subject to the inequality

$$c\bar{k}\mu_0 > \bar{\bar{k}}_0 = \max_{|\xi| \leqslant \bar{\delta}} \sup_{t \geqslant t_0} |ca_1 - a_2(1 + \xi) + (1 + \xi)^2/c| \tag{6.45}$$

for any controlled process $\boldsymbol{x}(t) \in G_{\bar{\delta}}$ the sign of $\dot{\lambda}(t)$ is determined by the sign of the last term in (6.44). This analysis also suggests that the controlled processes considered can oscillate about the domain $G_0 = \{\boldsymbol{x}: \sigma(\boldsymbol{x}) = 0\}$. To illustrate when a controlled process crosses

G_0, $|\lambda(t)|$ can increase from the value $|\lambda(t_1)| = 0$ at time t_1 of intersection to a value of $|\lambda(t_1 \pm \tau)|$ when according to the assumption made earlier the function $\sigma_\tau(t)$ changes sign. From the time instant $t = t_1 + \tau$ on, the function $|\lambda(t)|$ will fall off to the next zero, i.e. to the next crossing of G_0 by the process, and so on.

Deviation of controlled processes from the domain G_0 during such oscillations would be convenient to estimate with angular quantities. This purpose, in particular, is served by the quantity $\lambda(x)$. We wish to prove that given the inequality

$$\tau < \bar{\delta}/(c\bar{k}\mu_0 + \bar{\bar{k}}_0) \tag{6.46}$$

where k_0 is defined in (6.45), the considered processes $x(t) \in G_{\bar{\delta}}$ for all t.

We make use of equation (6.44) to estimate the largest "deviation" of controlled processes from the domain G_0. For definiteness, consider the following situation: let t_1 be a time instant such that for t in $(t_1 - \tau, t_1]$, $x(t_1) \in G_0$ and $\operatorname{sgn} \sigma_\tau(t) < 0$. Then from (6.44) and (6.45) it may be readily seen that $\lambda(t)$ increases on the interval $[t_1, t_1 + \tau]$ according to the equation

$$\lambda(t) = \int_{t_1}^{t} \left\{ \left[ca_1 - a_2(1 + \lambda \operatorname{sgn} x) + \frac{(1+\lambda \operatorname{sgn} x)^2}{c} \right] \operatorname{sgn} x + c\bar{k}\mu_0 \right\} d\tau \tag{6.47}$$

Let t' be the first time instant for the considered process to reach the boundary of the domain $G_{\bar{\delta}}$ and $t' \in [t_1, t_1 + \tau]$, then $\lambda(t') = \bar{\delta}$. Observing (6.47) this leads us in view of (6.45) to

$$\bar{\delta} \leqslant (c\bar{k}\mu_0 + \bar{\bar{k}}_0)(t' - t_1)$$

This inequality and (6.46) readily yield then that $\tau < t' - t_1$ which contradicts to our assumption that $t' \in [t_1, t_1 + \tau]$. Hence, for all t in $[t_1, t_1 + \tau]$ and subject to the aforementioned conditions, $\lambda(t) < \bar{\delta}$. Since given (6.45) $\lambda(t) > 0$ on the interval $[t_1, t_1 + \tau]$, then $\sigma_\tau(t) > 0$ for $t > t_1 + \tau$ and in agreement with (6.44) $\lambda(t)$ monotonously decreases for $t > t_1 + \tau$ down to a successive sign reversal of $\sigma_\tau(t)$. At a certain time instant t_1' the function $\lambda(t)$ will fall off to zero and the repetition of the above argument yields that for the process concerned the estimate $|\lambda(t)| < \bar{\delta}$ is valid for all $t \geqslant t_1$. Hence for this process we have $|\sigma[x(t)]| \leqslant \bar{\delta}|x(t)|$, $t \geqslant t_1$, equivalent to $x(t) \in G_{\bar{\delta}}$ for all $t \geqslant t_1$.

The above arguments prove the fact that any controlled process, once it has reached the boundary of the domain $G_{\bar{\delta}}$ from the areas $G_{\bar{\delta}}^{\pm}$ (see Fig. 6.10), remains in $G_{\bar{\delta}}$ and oscillates about the domain G_0 provided the inequalities (6.45) and (6.46) are satisfied and on the interval at least as long as τ preceding the process arrival at the $G_{\bar{\delta}}$

boundary the function $\sigma_\tau(t)$ has a sign coinciding with that of $\sigma(t)$, i.e. on this interval $\sigma_\tau(t) < 0$ when $x(t) \in G_{\bar{\delta}}^{=}$, and $\sigma_\tau(t) > 0$ when $x(t) \in G_{\bar{\delta}}^{+}$.

Let us now derive sufficient conditions for the function $\sigma_\tau(t)$ to take on the aforementioned signs. We note that if a controlled process $x(t)$ does not belong to $G_{\bar{\delta}}$ for t in $[t_0, t_0 + \tau]$, then $\operatorname{sgn} \sigma_\tau(t) = \operatorname{sgn} \sigma(t)$ for $t > t_0 + \tau$, and process behavior is defined by equation (6.35) for $t > t_0 + \tau$. Consequently, the behavior of this process may be analyzed with the use of the argument from the preceding section and, once (6.45) and (6.18b) are met, either $\| x(t) \| \to 0$ as $t \to \infty$ (and then we have $x^{II}(t)$ processes), or there exists a finite time t_1 when $x(t_1) \in G_{\bar{\delta}}$ (and then we have $x^{I}(t)$ processes studied above).

In summary, when the aforementioned conditions are satisfied, two outcomes are possible for the controlled processes. Some of them, beginning from a finite time instant, find themselves immersed in the domain $G_{\bar{\delta}}$ to remain therein afterwards, while the others converge to zero. It is an easy matter to verify that the processes $x(t) \in G_{\bar{\delta}}$ also converge to zero. To demonstrate, we invoke, like in the previous section, the differential constraint (6.30) to establish the validity of the estimate (6.32) for processes $x(t) \in G_{\bar{\delta}}$ for $t \geqslant t_1$; this estimate is as follows:

$$\| x(t) \| \leqslant \frac{1+c+\bar{\delta}}{c} \| x(t_1) \| \exp\left[-\frac{1-\bar{\delta}}{c} (t - t_1) \right]$$

As a consequence, with $0 < \bar{\delta} < 1$ we obtain for such controlled processes the desired equality $\lim\limits_{t \to \infty} \| x(t) \| = 0$.

It will be rather easy to estimate the maximum admissible dead time in switching at which all controlled processes converge to zero. We notice that, because $\bar{\delta} < 1$ the asymptotic solution of the control problem with the considered control algorithm and in the framework of the outlined formalism may be ensured for any τ satisfying, in accord with (6.46), the inequality

$$\tau < 1/(c\bar{k}\mu_0 + \bar{\bar{k}}_0) \tag{6.48}$$

Let us briefly discuss how increasing gain in the CFB loop influences this result. From (6.48) it can be seen that when one of the coefficients $\bar{k}$ or μ_0 increases, the range of admissible dead time decreases. This in particular leads us to conclude that for the ultimate gain control system, phase portrayed in Fig. 6.9, the considered control problem has no solution. It is also a known fact (see, e.g.

Refs. 1, 2, 6) that if the linear CFB of constant delay τ

$$u(t) = k\sigma_\tau(t)$$

$$\sigma_\tau(t) = -[c\dot{x}(t-\tau) + x(t-\tau)]$$

is used in place of the linear CFB (4.9) for $\mathcal{P}_x$-process control, then as the gain k goes to infinity—this is necessary to draw the behavior of controlled processes as close as possible to the specified performance—the respective closed-loop control system will always be unstable however small the dead time $\tau > 0$. Basing on this fact and the above development we may state that in the presence of time lag in obtaining $\sigma(t)$ and under limited information on the system state variables and parameters, for a given dependence of the controlled processes on the $\mathcal{P}_x$ parameters, the asymptotic solution of the control problem can be achieved with binary control systems having, as a necessity, a constraint in the COFB loop.

Earlier we have proved the convergence of controlled processes to the point O_x. To prove that O_x is asymptotically stable in the large we invoke the argument similar to that used in Section 6.3 for deriving the estimate (6.33). It is an easy matter to see that for the S_μ systems considered the bound (4.4a) is valid for all $t \geqslant t_0$, i.e. $\| \boldsymbol{x}(t) \| \leqslant N_\tau \| \boldsymbol{x}(t_0) \|$, at some constant N_τ.

The foregoing consideration, therefore, substantiates the following property:

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\mu}\{\mathcal{F}_{\text{pc}}, I_2; N_\tau\} \tag{6.49}$$

The controlled processes of first type obey an exponential bound with the exponent $-(1-\bar{\delta})/c$.

It will be noted that (6.49) differs from (6.3b). Comparing (6.49) with the respective expression for the on-off control system with $\mathcal{A}_\tau$ control algorithm, namely $\mathcal{P}_x \in \mathcal{C}_{(\Delta, \mathcal{A}_\tau)}\{\mathcal{F}_{\text{pc}}, I_2; N\}$, leads us to conclude that the dead time in on-off switching variously affects the behavior of controlled processes, depending on which loop contains this two-position element—the CFB or COFB loop. In the first case, the control problem is solvable Δ-accurately, in the second case it is solvable asymptotically and for certain controlled processes an exponential decline in the norm is ensured.

Let us examine now how the limitation on control inserted in the COFB loop affects the validity of the qualitative conclusions we have drawn above on the effects the time lag exerts on the performance of two-position control systems and binary control systems with bang-bang COFB. Assume that the aforementioned constraint on control is introduced, as before, by

$$u = M \operatorname{sat}[k(\mu)\, x/M] \tag{6.50}$$

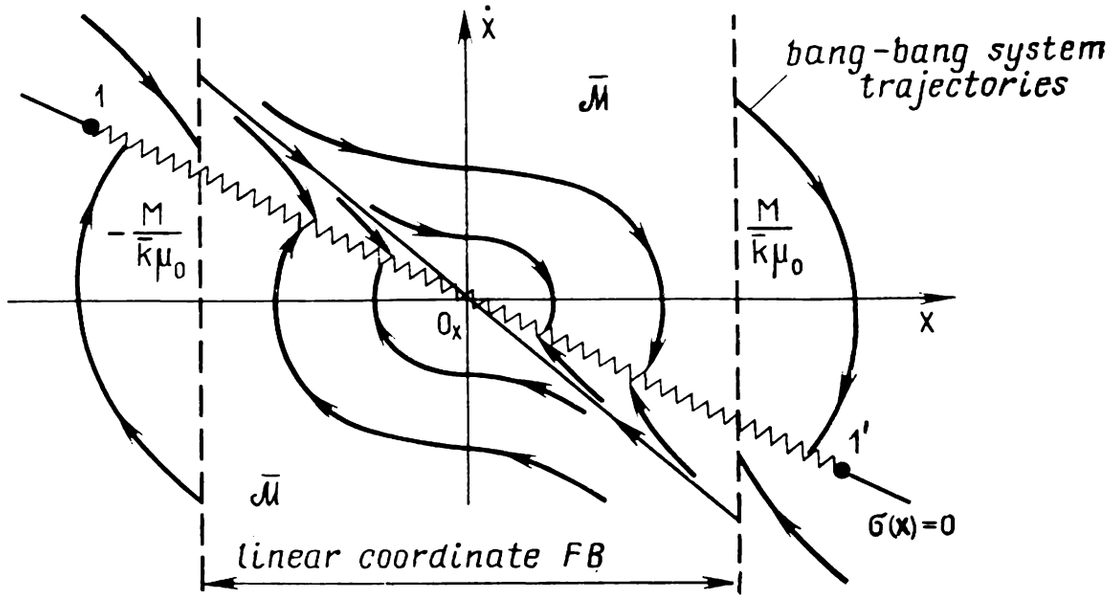

Figure 6.11

where M is a positive constant, and sat $(\cdot)$ is the saturation function defined in (6.12).

When $|k(\mu)x| \leqslant M$ and therefore $u = k(\mu)x$, we say about a zone of *linear* coordinate feedback. The respective domain on the phase plane is denoted by $\bar{\mathscr{M}}$. When $|k(\mu)x| > M$, we have the bang-bang control $u = M \operatorname{sgn}[k(\mu)x]$ in the CFB loop, so the behavior of controlled processes in this closed-loop control system is defined by the equations of the two-position control system (5.6).

Consider the situation with the dead time $\tau = 0$ in the two-position COFB S_μ system. Then in the zone of linear CFB the controlled processes are defined by equation (6.35). Consequently, for every fixed values of $\mathscr{P}_x$ parameters a_1 and a_2 from (4.2) and subject to (6.18) the qualitative phase portrait looks as indicated in Fig. 6.11. Points 1 and 1′ in this figure limit the sliding mode domain in the two-position control system (5.6).

Analysis of Fig. 6.11 suggests that in the S_μ system considered the properties of controlled processes become independent of the $\mathscr{P}_x$ parameters a_1 and a_2 only from the moment they arrive at the segment [1, 1′] of the straight line $\sigma(x) = 0$, more specifically, from the moment the control processes appear in the domain $\mathscr{M}$ defined in Section 5.3 as

$$\mathscr{M} = \left\{(x, \dot{x}): |x| \leqslant \frac{Mc^2}{\sup\limits_{t \geqslant t_0} |1 - ca_2(t) + ca_1(t)|}, \; c\dot{x} + x = 0\right\}$$

It will be noted that, unlike the two-position control system, now there occur controlled processes ($x^{II}(t)$ processes to be particular) that fail to reach $\mathscr{M}$ in a finite time. Observing this remark and assuming the viewpoint of independence of the motions of the plant parameters it may be said that the behavior of the S_μ system considered

and that of the two-step system coincide and, consequently, it would be wise to conclude that the constraint on control in the CFB loop prevents the construction of a control system with motions independent of a_1 and a_2 over the entire state space, as this is the case in the control system investigated in Section 5.2 (see Figs. 5.3, 5.4, and 6.9). This implies that the property expressed by (6.3b) is in principal unattainable in real systems and at best we may count on the achievement of the property

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}} \{\mathcal{F}_c,\ I_2;\ N,\ 1/c\}_{\mathcal{M}} \tag{6.51}$$

In the S_μ system under consideration, this property does not take place because here the control action $u\ (t) \in \mathcal{F}_s$, whereas (6.51) requires that $u\ (t) \in \mathcal{F}_c$. Comparing Figs. 5.6, 6.8, and 6.11, and the findings of the analysis carried out in the foregoing sections, it is an easy matter to prove for the S_μ system the validity of the expression

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\mu} \{\mathcal{F}_s,\ I_2;\ (1+c)/c,\ 1/c\}_{\mathcal{M}} \tag{6.52}$$

which took place for the bang-bang control law. It is in this sense that we have said above of there being no difference in the behavior of the S_μ system with two-step COFB and the system with two-step CFB.

Yet some difference is there and it is most pronounced with account of dead time in on-off switching. Below we demonstrate this statement. Let the time lag τ be other than zero and satisfy the inequality (6.46) at some constant $\bar{\delta} > 0$, (6.45) is met for $\bar{k}\mu_0$, and (6.18b) is valid. Then, for $x\ (t) \in \overline{\mathcal{M}}$ the controlled processes in the S_μ system concerned are described in (6.40), and for $x\ (t) \notin \overline{\mathcal{M}}$ by equation (5.6) where the function $\sigma_\tau\ (t)$ from (6.40) must be substituted for $\sigma\ (t)$.

Again we limit ourselves to considering those controlled processes for which phase representation of motion is feasible. Using the argument exploited in the preceding sections, it will be an easy matter to verify that for every fixed $\mathcal{P}_x$ parameters from (4.2) the qualitative behavior of such controlled processes is consistent with the phase portrait displayed in Fig. 6.12.

The dashed lines $\sigma'\ (x) = 0$ and $\sigma''\ (x) = 0$ represent the boundaries of sign reversal for the function $\sigma_\tau\ (t)$ for the two-position control system, while the chained lines within the domain $\overline{\mathcal{M}}$ represent the boundaries of the domain $G_{\bar{\delta}}$, i.e. the domain where the controlled processes oscillate about the line $\sigma\ (x) = 0$ in the S_μ system (6.40).

This figure clearly indicates that when the constraint (6.50) is

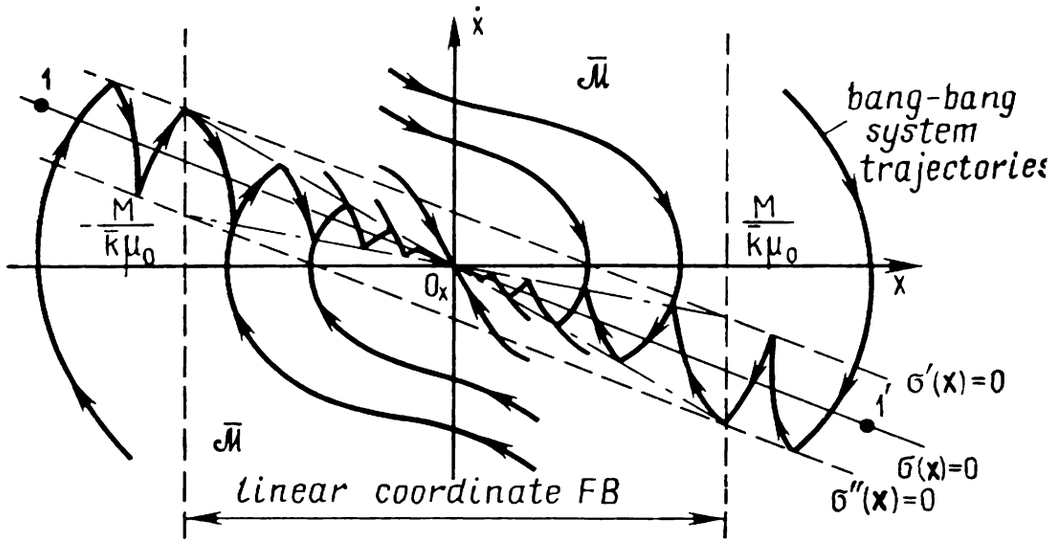

Figure 6.12

introduced in the CFB loop of the S_μ system with the bang-bang COFB and dead time, the possibility of a stable asymptotic solution to the control problem remains. Consequently, the earlier qualitative conclusion on the different effects of dead time on S_μ systems with a constraint in the COFB loop and on the two-step systems remains valid also when additionally a constraint is imposed on the control in the CFB loop.

A similar conclusion may be drawn for situations with other types of dynamic nonlinearity present in the control system. The respective substantiation is omitted here, but in our later consideration of some control systems the effect of inertial factors unaccounted for in the $\mathcal{P}_x$-process model will be treated in depth.

In what follows we shall provide an interpretation to the findings of the preceding paragraphs, pinpointing the different roles the constraints in the CFB and COFB loops play in solving the control problem.

High-gain systems are inherently unstable in the presence of dynamic nonlinearities. Insertion of a constraint on the control signal may convert them into dissipative systems (with Δ-accurate control). Changing to the binary, i.e. nonlinear, control systems preserves the "dissipative behavior" if there is no restriction in the COFB loop. It is only with a restriction inserted in the COFB loop. that the control problem can be solved asymptotically under the considered conditions. A constraint in the COFB loop may therefore be viewed as a means, maybe not unique, enabling an asymptotic solution for the control problem posed by parametrically ill-defined dynamic systems with dynamic nonlinearities.

In the automation of actual physical processes, the quality of control is always affected, to various extent, by the constraints imposed on control and gains, and the simplifications allowed in the

mathematical model of the process, therefore the conclusion drawn above is of principal significance. It implies that for a real process the control system designer should prefer nonlinear control laws. It is important then to identify where in the dynamic system the nonlinearities occur—for example, due to two-position elements, as was the case in the earlier considered control systems—and what channel will be used for derivative control action. In other words, primary importance for the quality of control is associated now with the structure of the nonlinear controller and the channels it defines for processing the information used for control. It should be expected that a change in the control objective, say changing the requirement for an asymptotic solution to a Δ-accurate solution, may call for a considerable alteration in the structure of the controller. The theory of binary automatic control systems is an attempt to develop a practical tool for handling these problems.

Summing-up reappraisal. Let us sum up the foregoing investigation and lay out the plan of our later discussion. Imagine that for an ill-defined $\mathcal{P}_x$-process, whose parameters vary arbitrarily over wide ranges, it is required to provide asymptotically accurate control and to ensure independence of the controlled processes in the closed-loop control system of the $\mathcal{P}_x$ parameters. In practical applications this problem is aggravated by many a factor, such as constraints on the control signal and gains, assorted nonlinearities or inaccuracies in the data on the system status. These factors must be allowed for as they markedly affect the behavior of a real control system.

When taken into account in a control system design these factors affect the choice of a feasible control algorithm. For example, the methods of linear control theory are not always useful in handling the control problem on hand, and under the specified conditions it is advisable to stick to nonlinear control algorithms. Unfortunately, no regular method so far exists for setting up nonlinear control systems. The theory of binary control systems views as one of its goals the development of methods of synthesis of nonlinear control algorithms which would ensure high performance in handling ill-defined dynamic processes.

Inserting a limiting element into the COFB loop of a binary control system automatically bounds the CFB loop gains, thus setting the framework for solving the control problem under a constraint on the control in the CFB loop and in the presence of dynamic nonlinearities.

Analysis of binary control systems with time invariant COFB has indicated that the complete solution of the control problem is unattainable over the class of continuous controls. In addition, such control systems contain controlled processes (x^{II} processes) which present certain difficulties in establishing their dependence on the $\mathcal{P}_x$ parameters. In this connection, binary control systems with

dynamic COFB laws and other types of feedback are worth consideration. Our further development will be devoted to the possibilities offered by this formalism.

To conclude this section, two more remarks are in order. Since with the $k(\mu)$ dependence (6.6) in the CFB loop the limit value μ_0 in the COFB loop can always be accounted for in assigning the gain $\overline{k}$, for convenience we shall henceforth assume that $\mu_0 = 1$ and $\overline{k} = k^0$. Finally, we have established earlier that with a constraint (6.50) on control in the CFB loop, the behavior required of the closed-loop control system, as written in (6.51), can be provided only within a bounded domain (domain $\mathcal{M}$ defined in Section 5.2). In our subsequent consideration we take this fact into account, yet omit this domain symbol in the respective formulas and avoid special mention of this fact.

6.6 Integral Coordinate-Operator Feedback

Embarking on the dynamic laws for COFB we immediately have to decide which of these laws is to be considered first. Obviously, the choice of a control law should be governed by its efficiency in the achievement of the desired properties. We take up the integral laws of GOFB first. The underlying cause for this choice will be given below.

Inserting the integral of error into the feedback law of a linear control system endows the system with the property of asymptotic control (or "astatism" if in the Russian literature) when subject to a step input $y^r(t)$ in the block diagram of Fig. 2.5. Strictly speaking, this effect takes place only for stationary processes. If the controlled process is nonstationary and the reference input $y^r(t)$ is a variable but bounded function of time, integral feedback provides only a Δ-accurate control (or "dissipatism" if in the Russian literature on the subject) for the closed-loop control system.

It will be recalled that asymptotic control implies that the controlled processes $x(t)$ converge to the origin of the error space (phase or x space), whereas the Δ-accurate control implies that they converge to a Δ-neighborhood of the origin of the phase space. These facts may be stated in another way, namely, parametric disturbances in the controlled plant prevent it from being asymptotically controlled with the help of ordinary integral feedback when subjected to a step reference input. In this connection we may reformulate the problem to aim it at deriving control laws ensuring asymptotic control in this case as well. This problem is solvable in the class of binary control systems.

Let us look at another aspect of this problem. In handling parametrically ill-defined processes, say $\mathcal{F}_x$-processes, the control engi-

neer strives to ensure invariable transition processes. This objective is central to this work as well. It will be achieved provided the controlled process, beginning from a certain time instant, is governed by a differential equation whose parameters can be specified at will — i.e., they are circuit parameters of the control system—and independently of the controlled process parameters. This property is sometimes read as the equations of motion of the closed-loop control system being independent of, or invariant to, parametric disturbances.

For such a differential equation, often referred to as the *reference equation*, we introduce a space of parameters, the latter called the π space. In this space a fixed point corresponds to a desired transient response. For instance, the examples taken up earlier have $c\dot{x}(t) + x(t) = 0$ as the reference differential equation with $\pi^* = (1, c)$ being the corresponding point in the π space.

The π space formalism allows for some new concepts and definitions. When the controlled processes exactly satisfy the reference equation we commonly say their dynamics is independent of the controlled process parameters. When the behavior of controlled processes is governed by a differential equation close, in a certain sense, to the reference equation we say that their dynamics *weakly depends* on the process parameters (or that "parametrical dissipatism" takes place—if in the Russian literature on the subject).

By way of example, the controlled processes $x(t) \in G_\delta$ are governed by the differential equation

$$c\dot{x}(t) + [1 + \delta\xi(t)]\, x(t) = 0$$

where $\xi(t)$ is a function such that $|\xi(t)| \leqslant 1$.

In the π space to this equation for every t there corresponds a point $\pi^t = (1 + \delta\xi(t), c)$. The distance from this point to the point corresponding to the reference equation is $\|\pi^t - \pi^*\| = \delta |\xi(t)| \leqslant \delta$. For sufficiently small $\delta > 0$ these points lie close and we can say that the controlled process dynamics is close to the reference behavior. In the phase space, however, when the controlled processes go stable in a neighborhood of the desired point, the origin of this space, we say of the control system as being Δ-accurately controlled or exhibiting "dissipatism". It would be natural to refer to this "dissipative behavior" as being of coordinate origin. Accordingly, given the inequality $\|\pi^t - \pi^*\| \leqslant \delta$ is satisfied, it is appropriate to speak of the system exhibiting "parametric dissipatism". We recall that saying of a system being Δ-accurately controlled we imply an imprecise solution of the control problem, whereas being "parametrically dissipative" means an inaccurate achievement of the desired transient behavior.

Finally, if for $t \to \infty$ the parameters of the differential equation satisfied by the processes being controlled approach asymptotically the parameters of the reference equation, then we say that the controlled process dynamics is *asymptotically independent* of the plant parameters, or in the local parlance that "parametric astatism" takes place. To illustrate, if in the case considered above the function $\xi(t)$ is such that $\lim_{t\to\infty} \xi(t) = 0$, then $\lim_{t\to\infty} \| \pi^t - \pi^* \| = 0$ and "parametric astatism" takes place.

The development of parametrically "astatic" control systems may be viewed as a method of constructing high performance control systems for handling parametrically ill-defined processes. Indeed, when all the solutions to the reference equation converge to zero and the control system exhibits a parametrically "astatic" behavior, then the control problem is solved asymptotically, yielding a transient process dynamics close to and asymptotically approaching that specified.

From this aspect it would be of interest to evaluate the possibilities of binary control systems with an integral law of COFB. Our hopes for a positive effect in such systems are fed by the following argument. Bang-bang control systems are known to be amenable to asymptotic problem solving under limited coordinate disturbances represented, for example, by a bounded forcing function $y^r(t)$. For binary control systems with a bang-bang COFB law the controlled process behavior can be made independent under limited parametric disturbances. Changing from the bang-bang CFB law to an integral law opens the possibility of constructing "astatic" control systems. By analogy we expect that the substitution of an integral law for the bang-bang law in the COFB loop will yield control systems with parametrically "astatic" behavior in handling step variations in plant parameters. Another positive effect of this conversion is that the control $u(t)$ in such binary control systems becomes a continuous time function, as required in (6.51).

It is worth emphasizing that for all similarity of the problems of ensuring coordinate and parametrically "astatic" behavior, they are rather different. When parametric disturbances are handled, the difference is caused primarily by the principally nonlinear control laws and systems facing the designer.

In conclusion we notice that all the subsequent discussion will be devoted to the study of binary control algorithms which render a closed-loop control system parametrically "astatic", i.e. endow it with a response asymptotically independent of the plant parameters. It does not mean, of course, that control systems without this property are of no value. The thing is that parametrically "dissipative" systems, as a rule, are simpler and, therefore, more interesting for applications. This comparison is carried out not only because we

are interested in solving this problem per se. Looking ahead we may disclose that the desired solution will be achieved only with a configuration exploiting all types of feedback, namely, CFB, COFB, OFB, and OCFB.

6.7 Integral Coordinate-Operator Feedback with a Constant Integration Rate

In agreement with the adopted development plan, consider a binary control system having two types of feedback and integral COFB. According to our previous nomenclature, we will refer to all such algorithms of control for the $\mathcal{P}_x$-process as $\mathcal{A}_\mu$-algorithms, and the associated closed-loop control systems as S_μ systems. The transformation effected in the CFB loop will be defined as

$$u(t, x) = k(t)\, x \tag{6.53}$$

where $k(t)$ is a variable and bounded coefficient. It is variable by means of the operator variable $\mu(t)$ as

$$k(t) = k^0 \mu(t) \operatorname{sgn} x(t) \tag{6.54}$$

The operator variable $\mu(t)$ itself is formed in the COFB loop by an integral law and its differential form is

$$\begin{aligned} \dot{\mu}(t) &= -\alpha \operatorname{sgn} \sigma[x(t)] && \text{for } |\mu(t)| \leqslant 1 \\ &= -\omega\mu(t) && \text{for } |\mu(t)| > 1,\ |\mu(t_0)| \leqslant 1,\ t \geqslant t_0 \end{aligned} \tag{6.55}$$

Here as before $\sigma(x)$ represents the COFB loop error, defined in (5.1), namely, $\sigma(x) = -c\dot{x} - x$.

In (6.55) α and ω are some positive constants. The coefficients α and k^0 are circuit parameters of the control algorithm and can be selected at will; ω is an arbitrary, generally speaking, arbitrarily small number whose implication will be discussed below.

Expressions (6.53) and (6.54), equation (6.55) and the equation of the $\mathcal{P}_x$-process (Chapter 4) define the closed-loop S_μ system. We will refer to the entire set of these equations as the equations of S_μ system. Upon the substitution of (6.53) and (6.54) in the $\mathcal{P}_x$-process equation, the S_μ system is defined by equation (6.55) and the equation

$$\ddot{x}(t) + a_2(t)\,\dot{x}(t) + a_1(t)\,x(t) = -k^0\mu(t)\,|\,x(t)\,|, \quad t \geqslant t_0 \tag{6.56}$$

The control law of COFB (6.55) ensures a uniform variation of $\mu(t)$ for $|\,\mu\,| \leqslant 1$ and prevents it from leaving the interval $[-1, 1]$—a necessary condition to observe the constraint on the CFB loop gain, which in this case has the form $|\,k(t)\,| \leqslant k^0$. The subject of constraints has been discussed in depth in preceding sections and will

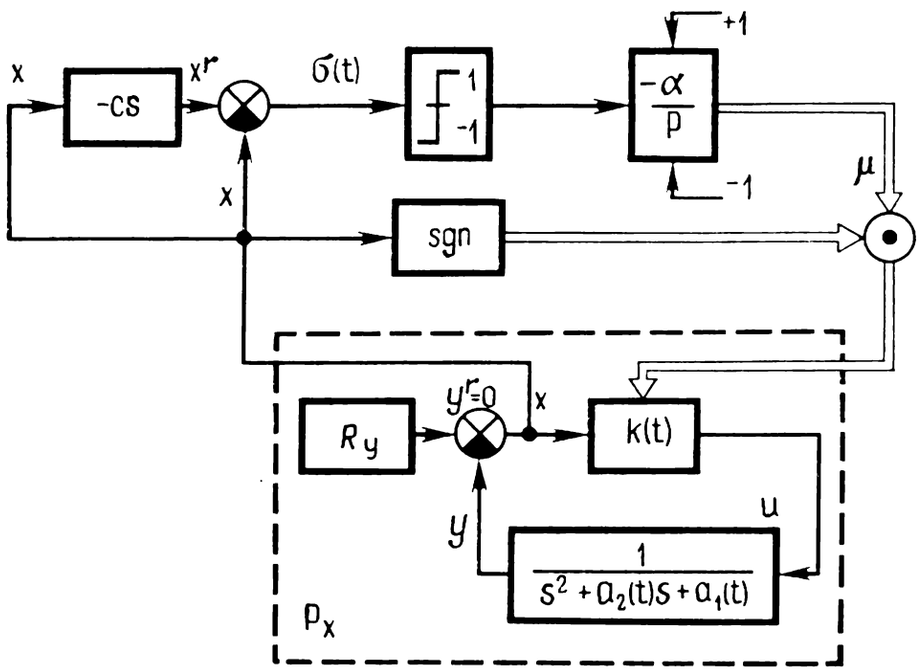

Figure 6.13

not be repeated here. It is worth recalling that such a COFB law forms feedback related to the class of admissible feedbacks defined in Chapter 6.

The block diagram associated with the S_μ system under consideration is shown in Fig. 6.13. The integral-like symbol represents a limiting element in the law (6.55) responsible for the formation of the operator variable $\mu\ (t)$.

From expressions (6.55) and (6.56) it follows that the S_μ system is described by differential equations with a discontinuous right-hand side. The discontinuities occur at $\sigma\ (x) = 0$ and $|\ \mu\ | = 1$. The solution to such an S_μ system will be understood in the sense of Filippov's work [93]. The topics of whether or not the solution exists and can be continued on the entire semiaxis $t \geqslant t_0$ will be treated here in agreement with the findings of this publication.

The state space for the S_μ system concerned is three-dimensional, $(x\ (t),\ \mu\ (t))$. Each solution is seen to consist of two functions: $x\ (t)$ is a continuously differentiable vector function for all $t \geqslant t_0$ (controlled process), and $\mu\ (t)$ is a scalar function, absolutely continuous, after Ref. 93, for all $t \geqslant t_0$. Particular solutions are determined by specifying the initial pair $(x\ (t_0),\ \mu\ (t_0))$. We will consider only those solutions which are generated by the initial pairs from the set $V = \{(x,\ \mu)\colon x \in \mathbb{R}^2_x\ |\ \mu\ | \leqslant 1\}$, i.e. for $(x\ (t_0),\ \mu\ (t_0)) \in V$. This fact is indicated in equation (6.55) by the initial condition for $\mu\ (t)$. We will refer to such solutions of S_μ systems as V solutions.

From (6.55) it will be seen that any V solution remains belonging to this kind for all $t \geqslant t_0$, and any other solution becomes a V solution in a finite time interval. Accordingly, in what follows we may limit ourselves to analyzing V solutions only.

Let us have a closer look at the solutions defined by (6.55) and (6.56). We will aim our evaluation at satisfying, for some choice of circuit parameters in such systems, the collection of requirements contained in (6.51) and ensuring for asymptotic independence of plant parameters for system response.

The general scheme of the investigation coincides, on the whole, with that we have employed above in the analysis of S_μ system with static COFB. More specifically, it consists in securing the conditions for all controlled processes to be embedded, from some time instant, into the domain $G_{\bar{\delta}}$ defined by (6.41) to remain there for all subsequent t. Then at sufficiently small value of $\bar{\delta}$ the behavior of the controlled processes is close to the performance specifications.

The higher dimensionality of S_μ systems with integral COFB brings about a number of salient features in the investigation. Besides, some concepts need refinement. The principal findings of the evaluation will be given in the form of statements which will be conveniently formulated with the terms introduced below. It will be recalled that now any feedback system, S_c, under examination will be denoted by the symbol S with a subscript indicating the type of feedback. The definitions outlined here will be used for closed-loop control systems with different types of feedback. The first definition gives an exact specification of what is meant under the convergence of controlled processes to some domain $G_{\bar{\delta}}$ given over the phase space.

Definition 6.7-1. The set $G_{\bar{\delta}}$ is called a domain of attraction, or simply attracting, for an S_c system if for any V solution $(\boldsymbol{x}(t), \mu(t))$ of this system such that $\boldsymbol{x}(t_0) \notin G_{\bar{\delta}}$ there exists a finite time instant t_V such that either $\boldsymbol{x}(t_V) \in G_{\bar{\delta}}$ or $\lim_{t\to\infty} \boldsymbol{x}(t) \in G_{\bar{\delta}}$.

Because in S_μ systems with dynamic COFB laws, $\boldsymbol{x}(t)$ represent only a part of system state variables—$\boldsymbol{x}(t)$ is a projection of V solutions onto the phase space—the domain $G_{\bar{\delta}}$ in such systems is, as a rule, not invariant as was the case with S_μ systems having static COFB laws. It appears, however, that within $G_{\bar{\delta}}$ some subdomains can be singled out so that V solutions taking off from such subdomains fail to leave $G_{\bar{\delta}}$ at all t. Therefore the concept of invariance of $G_{\bar{\delta}}$ for the S_μ systems concerned should be replaced with some other concept. Referring to Fig. 6.14, assume that G is a subdomain of $G_{\bar{\delta}}$ and give the following definition.

Definition 6.7-2. The domain G_δ is called *G-invariant* for an S_c system at $t \in [t_1, t_2]$, $t_2 \geqslant t_1$, if for its any V solution such that $\boldsymbol{x}(t_1) \in G$ we have $\boldsymbol{x}(t) \in G_\delta$ for all $t \in [t_1, t_2]$.

Remark. For the case of $t_1 = t_0$ and $t_2 = \infty$, the time interval $[t_0, \infty)$ will be not indicated in the G invariance designation.

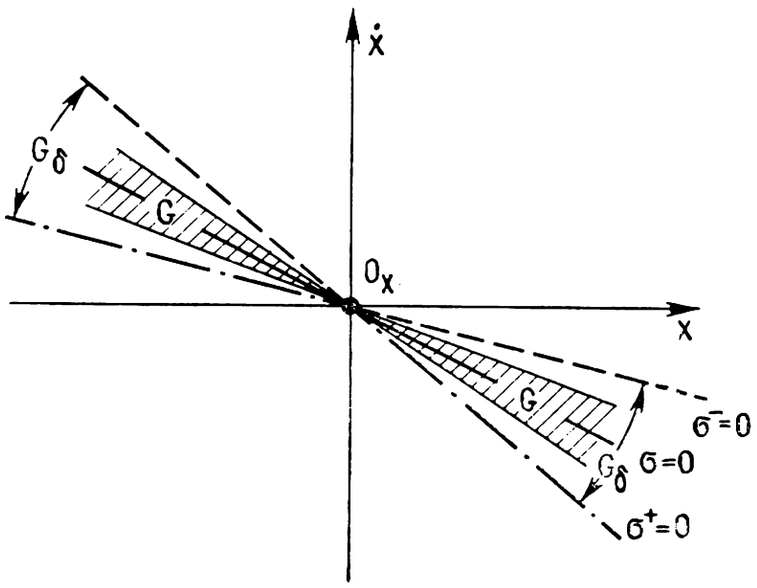

Figure 6.14

An immediate implication of this definition is that the case of $x(t) \in G_{\bar{\delta}}$ and, hence, a close dynamic behavior of the controlled processes to the desired performance will take place provided the domain $G_{\bar{0}}$ is G-invariant and the controlled processes find themselves in the domain G.

For S_μ systems with integral COFB, it would be convenient to take the domain $G_0 = \{x: \sigma(x) = 0\}$ for the domain G, therefore it would be sensible to define the conditions under which G_0 is a domain of attraction for S_μ systems. This problem may be reduced to a similar problem for VS systems. Indeed, if the controlled process $x(t)$ is such that on a time interval at least $2/\alpha$ long the function $\sigma(t) = \sigma(x(t))$ is other than zero, then in agreement with (6.55) $\mu(t)$ at the end of this interval takes on the value $\mu(t) = -\text{sgn}\,\sigma(t)$. Substituting this expression in (6.56) yields an equation describing a VS system. If the domain G_0 is attracting for such a VS system, it will be attracting for the S_μ system considered as well.

The following definition introduces a concept of a somewhat stronger kind than that of a domain of attraction. This rigour facilitates the formulations and the proofs of statements given below.

Definition 6.7-3. The domain G_0 is called Θ-attracting for a VS system if for any controlled process $x(t)$ such that $x(t_0) \notin G_0$ there exists a finite time instant t' when $x(t') \in G_0$ or else $\lim_{t\to\infty} \|x(t)\| = 0$.

It should be noted that the conditions of Θ-attraction for VS systems are well documented in the literature on variable structure system, e.g., see [24, 25]. We shall employ some of these conditions in the ensuing consideration.

The study of S_μ systems with integral COFB in this section and binary systems appearing later is carried out with a uniform procedure. The ultimate goal of each system analysis is to reveal whether or not system's behavior is consistent with the performance specifications contained in the definition of the complete solution to the control problem. The investigations will be qualitative in nature and consist of solving a set of principal problems as follows.

Problem A is concerned with the asymptotic behavior of controlled processes $x(t) \in G_\delta$ where G_δ is the domain defined by (6.24).

In most cases this problem is solvable with the help of the differential constraint $\dot{x} = -x/c - \sigma/c$ and the inequality $|\sigma| \leqslant \delta |x|$ valid subject to $x(t) \in G_\delta$. We remark that the controlled processes in this problem were referred to above as x^{I} processes as for the constant parameter $\delta > 0$ being sufficiently small the behavior of x^{I} processes is close to the desired performance corresponding to the value $\delta = 0$.

Problem B establishes the property of G invariance of the domain G_δ.

This problem occurs from the fact that for binary control systems with dynamic COFB laws the domain G_δ is, as a rule, no longer invariant. Accordingly, it is required to determine a domain $G \in G_\delta$ which would ensure a conditional invariance of G_δ; this was defined above as G invariance. It is worth noting that this problem is more intimately connected with the control algorithm used and its solution establishes the principal relationships necessary for selecting the circuit parameters of the system.

Problem C determines conditions for the domain G_δ to be attracting.

The solution of this problem enables all the controlled processes to be divided in two families: $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. The former reach G_δ in a finite time interval (more accurately, they reach G) and remain there (if Problem B is solved) subsequently. The latter satisfy the inequality $\delta |x(t)| > |\sigma(t)|$, i.e. $x^{\mathrm{II}}(t) \notin G_\delta$, beginning from a certain time and the thing that matters is that in the limit these processes converge to a certain point of G_δ.

Problem D is to study the asymptotic behavior of x^{II} processes and establish conditions when $\lim_{t \to \infty} \| x^{\mathrm{II}}(t) \| = 0$.

The solution to this problem, along with the solution of Problem A, warrants the convergence of all controlled processes in the system to the origin of the phase space.

Problem E evaluates the behavior of controlled processes in the vicinity of the point O_x.

The solution of this problem establishes the stability of this point, or provides an bound of the neighborhood where all controlled processes go stable.

Problem F is to investigate the dependence of controlled process behavior on the $\mathcal{P}_x$ parameters and reveal how much this behavior is influenced by the system's circuit parameters.

The major objective of the study in this case is to establish whether the system is "parametrically astatic" or "dissipative", i.e. whether the system response is asymptotically independent of, or weakly depends on, plant parameters. The properties of controlled processes established in this procedure are specific for each type of binary control algorithm.

In what follows we will refer to the solution of the aforementioned set of problems as the *qualitative investigation* of the control system. For binary control systems with dynamic COFB laws, such an investigation is sensible if there is no way to maintain the equality $\sigma(t) = \sigma(x(t)) = 0$ exactly beginning from a certain time instant. Above we have noticed this fact, now we wish to prove it.

Let us put down the expression for $\dot{\sigma}(x(t))$. Omitting the algebraic manipulations involved and the argument t for the time functions we reproduce the final expression for this derivative

$$\dot{\sigma} = -\frac{ca_2 - 1}{c}\sigma - \left(\frac{ca_2 - 1}{c} - ca_1\right)x + ck^0\mu|x| \tag{6.57}$$

The right-hand side of this expression is piecewise continuous and therefore the identity $\sigma(t) = 0$ holds on a certain time interval provided that

$$\mu = \frac{1}{ck^0}\left(\frac{ca_2 - 1}{c} - ca_1\right)\operatorname{sgn} x$$

The $\mathcal{P}_x$ parameters a_1 and a_2 are, however, unknown and vary with time, therefore such an equality, in general, cannot be effected with a COFB law such as (6.55) at a finite α at all moments of time. Hence, there is no way of keeping the equality $\sigma(t) = 0$ in this S_μ system and therefore it makes sense to speak of its approximation. In this text, this approximation is understood as satisfying the inequality $|\sigma(t)| \leqslant \delta |x(t)|$ at a sufficiently small positive constant δ. It will be noted that this inequality is equivalent to the fact that $x(t) \in G_\delta$. If the parameter δ is allowed to be selected at will, then assuming it being sufficiently small we can draw the behavior of the controlled processes close to the required performance outlined in (6.51). Having finished with these remarks we go over to a qualitative investigation of the S_μ system with integral COFB.

Problem A. Recall that it is concerned with the convergence of the controlled processes $x(t) \in G_\delta$ to the point O_x. As mentioned above, in $\mathcal{P}_x$-process control this problem is solved with the help of the differential constraint $\dot{x} = -x/c - \sigma/c$ and the inequality $|\sigma| \leqslant \delta |x|$. Actually, we have already tackled such problems, see, e.g., expressions (6.31) and (6.32). Since we shall make numerous

references to the respective result, for convenience it will be reproduced in the form of a statement as follows.

Theorem 6.7-1. Let $x(t) \in G_\delta$ for a V solution $(x(t), \mu(t))$ for t in $[t_1, t_2]$, $t_2 > t_1 \geqslant t_0$. Then for this V solution at $t \in [t_1, t_2]$ and a constant $c > 0$

$$\| x(t) \| \leqslant \frac{1+c+\delta}{c} \| x(t_1) \| \exp\left[-\frac{1-\delta}{c} (t - t_1) \right] \tag{6.58}$$

The proof of this theorem is based on the Grownwall-Bellman inequality and is actually carried out in Section 6.3. The inequality (6.58) comprising the key subject of Theorem 6.7-1, aids in establishing the convergence to O_x of those controlled processes which belong to G_δ.

Corollary 6.7-1. Let under the conditions of Theorem 6.7-1 $t_2 = \infty$ and

$$0 < \delta < 1 \tag{6.59}$$

then $\lim\limits_{t \to \infty} \| x(t) \| = 0$.

The proof for this corollary immediately follows from the bound (6.58) subject to (6.59).

For the S_μ system under investigation, the controlled processes converge to O_x if $x(t) \in G_\delta$. Conditions for this to take place are outlined below.

Problem B. Because for the S_μ system it is allowable to take the domain $G_0 = \{x: \sigma(x) = 0\}$ for the domain G involved in the definition of G invariance for G_δ, the sufficient conditions for $x(t) \in G_\delta$ are given by the following statement.

Theorem 6.7-2. Let for the S_μ system described by equations (6.55) and (6.56) be given $0 < \delta < 1$, $c > 0$,

$$k^0 > \sup_{t \geqslant t_0} \left| a_1 - a_2 \frac{1+\delta\xi}{c} + \left(\frac{1+\delta\xi}{c} \right)^2 \right| \tag{6.60}$$

for all $|\xi| = 1$, and

$$\alpha \geqslant \frac{2c}{\delta} \left[k^0 + \sup_{t \geqslant t_0} \left| a_1 - a_2 \frac{1+\delta\xi}{c} + \left(\frac{1+\delta\xi}{c} \right)^2 \right| \right] \tag{6.61}$$

for all ξ with $|\xi| \leqslant 1$. Then the domain G_δ is G_0 invariant for this S_μ system.

This theorem establishes the lower bounds for the gains in the CFB and COFB loops such that retain every controlled process once appeared in G_0 (i.e. on the line $\sigma(x) = 0$) in the domain G_δ in all subsequent variations. It is in order to remark that the right-hand sides in (6.60) and (6.61) can be readily computed once the bounds for the variation ranges of $\mathcal{P}_x$ parameters a_1 and a_2 are given. Now we present the complete proof for this theorem.

Proof. Consider a V solution $(x(t), \mu(t))$ of the S_μ system (6.55) and (6.56) for $t \geqslant t_0$. Let at a time instant $t_1 \geqslant t_0$ for this solution $\sigma(x(t_1)) = 0$ as required in the theorem statement. Examine the subsequent behavior of such a solution. If at $t \in [t_1, t_1 + \Delta]$ (Δ is a positive number) this V solution has the form $(O_x, \mu(t))$, where $\mu(t)$ is any absolutely continuous function with $|\mu(t)| \leqslant 1$ and almost everywhere in the interval $[t_1, t_1 + \Delta]$ there holds $|\dot{\mu}(t)| \leqslant \alpha$, then this is an ordinary solution understood in Filippov's sense [93], which according to the equation of motion for S_μ system preserves the form $(O_x, \mu(t))$ for all $t \geqslant t_1$. Because $O_x \in G_\delta$ this implies that for the V solution examined $x(t) \in G_\delta$.

Consider now an arbitrary V solution $(x(t), \mu(t))$ for which $\sigma(x(t_1)) = 0$, but now $x(t_1) \neq O_x$, then it is quite obvious that for this V solution $x(t) \neq O_x$ for all $t \geqslant t_1$. Assume that for the chosen V solution there exists a time instant $t' > t_1$ when the controlled processes reaches the boundary of G_δ, i.e. $x(t') \in \partial G_\delta$, where ∂G_δ is the boundary of G_δ defined as

$$\partial G_\delta = \{x : [\sigma^+(x) = 0] \cup [\sigma^-(x) = 0];\quad \sigma^+ = \sigma - \delta |x|,$$
$$\sigma^- = \sigma + \delta |x|\} \tag{6.62}$$

Under this assumption, prior to t' there exists a time instant $t_1'' \geqslant t_1$ such that $\sigma(x(t_1'')) = 0$ and the function $\sigma(x(t))$ retains its sign for t in $[t_1', t']$. Without any loss of generality let, for definiteness, $\sigma(x(t)) > 0$ for $t \in (t_1', t']$. We wish to demonstrate that $\mu(t') = -1$. Suppose it is not true, say $\mu(t') > -1$. Then from equation (6.55) of the S_μ system there follows $t' - t_1' < 2/\alpha$. Let us look, under this assumption, at the variation of $\lambda(t) = \sigma(x(t))/\delta |x(t)|$ along the V solution considered over the time interval from t_1' to t'. The function $\lambda(t)$ represents the instantaneous slope of the straight line passing through point $x(t)$ and the origin of the phase space. Given $x(t) \in G_\delta$, the function $\lambda(t)$, as follows from its definition, satisfies $|\lambda(t)| \leqslant 1$, with $|\lambda| = 1$ being valid only for $x \in \partial G_\delta$.

Let us convince ourselves that $\lambda(t)$ is defined and continuous on the considered time interval $(t_1', t']$. If for some $t'' \in (t_1', t']$ $x(t'') = 0$, then in view of $\sigma(t) < \delta |x(t)|$ we have also $\sigma(t'') = 0$ on this interval. Then, because $\sigma = -(c\dot{x} + x)$ and $\dot{x}(t'') = 0$, the V solution considered has the form $(O_x, \mu(t))$, but such solutions have already been examined in the first part of the proof and excluded by the initial assumption in the second part of the proof. Hence, observing $\lambda(t_1') = 0$, we get

$$\lambda(t) = \int_{t_1'}^{t} \frac{\dot{\sigma}(\tau) - \lambda(\tau)\delta |\dot{x}(\tau)|}{\delta |x(\tau)|} d\tau$$

or upon invoking the equation of S_μ system dynamics and some straightforward algebra

$$\lambda(t) = \int_{t_1'}^{t} \left\{ \frac{c}{\delta} \left[a_1 - a_2 \frac{1+\delta\lambda \operatorname{sgn} x}{c} + \left(\frac{1+\delta\lambda \operatorname{sgn} x}{c} \right)^2 \right] \operatorname{sgn} x + ck^0 \mu/\delta \right\} d\tau \tag{6.63}$$

Designate $\xi = \lambda \operatorname{sgn} x$. Observing that for $x \in G_\delta$ $|\xi| \leqslant 1$ we perform absolute value estimations in the integrand of (6.63) and recognizing (6.61) arrive at the inequality $\lambda(t) \leqslant (t - t_1')\alpha/2$ valid for all t in $[t_1', t']$. Since we have established above that $t' - t_1' < 2/\alpha$, then for all $t \in [t_1', t']$ $\lambda(t) < 1$ contrary to the assumption of $x(t') \in \partial G_\delta$ when $\lambda(t') = 1$. This contradiction proves that $\mu(t') = -1$.

Now let $x(t')$ be the value of the controlled process corresponding to the considered V solution belonging to the boundary of G_δ, i.e. $\sigma^+(x(t')) = 0$. Observing that $\mu(t') = -1$, as established above, we wish to demonstrate that such controlled process cannot leave G_δ.

The derivative of the function $\sigma^+(x(t))$ subject to $\sigma^+(x) = 0$ has the form

$$\dot{\sigma}^+ \mid_{\sigma^+=0} = c \left[a_1 - a_2 \frac{1+\delta \operatorname{sgn} x}{c} + \left(\frac{1+\delta \operatorname{sgn} x}{c} \right)^2 \right] x + ck^0 \mu |x| \tag{6.64}$$

Let $[t', t'']$ be a time interval such that for any t in this interval $\sigma(x(t)) > 0$. Then, in agreement with (6.55), $\mu(t) = -1$ on this interval. Observing this fact and the inequality (6.60) on the right-hand side of (6.64) we get $\dot{\sigma}^+ < 0$, hence the controlled process $x(t)$ cannot enter the domain $\sigma^+(x) > 0$ from the line $\sigma^+(x) = 0$ for $t \geqslant t'$. This implies that for all $t \geqslant t'$ $x(t) \in \{x: [\sigma(x) > 0] \cap [\sigma^+(x) \leqslant 0]\}$, so that for all $t \geqslant t'$ $x(t) \in G_\delta$, or there exists a time t_1'' when $\sigma(x(t_1'')) = 0$. Considering t_1'' as an initial time instant we repeat the above argument as many times as necessary. Therefore, under the conditions of the theorem $x(t) \in G_\delta$ for all $t \geqslant t_1$, and hence the domain G_δ is G_0 invariant. Q.E.D.

To summarize, given the inequalities stated in Theorem 6.7-2 are satisfied, every controlled process of S_μ system with integral COFB once finding itself in $G_0 = \{x: \sigma(x) = 0\}$ subsequently belongs to G_δ. If additionally the conditions of Theorem 6.7-1 and Corollary 6.7-1 are met, then these controlled processes converge to the origin of the phase space and the exponential estimate (6.58) is valid for these processes. In order that these facts take place, however, we need to ensure conditions warranting the arrival of controlled processes in G_0. These conditions are defined below.

Problems C and D. Here we establish conditions under which the domain G_δ is attracting for the S_μ system (6.55)-(6.56), x^{I} processes achieve G_0 in a finite time, and x^{II} processes asymptotically approach the point O_x. All results will be formulated in terms of Definitions 6.7-1 through 6.7-3 introduced above.

Let us evaluate first the conditions for G_δ to be a domain of attraction. For this purpose we use the equation of the VS system corresponding to the considered S_μ system (6.55)-(6.56); recall that the relation between these systems have been evaluated earlier. The equation of motion for the VS system is as follows:

$$\ddot{x}(t) + a_2(t)\dot{x}(t) + a_1(t)x(t) = k^0 \mid x(t) \mid \operatorname{sgn}\sigma(t), \quad t \geqslant t_0 \tag{6.65}$$

The theorem to be given below establishes a relationship between the $\mathcal{O}$-attraction of G_0 for the VS system and the G_δ-attraction for the S_μ system thereby reducing the solution of the G_δ-attraction problem to a simpler problem of the $\mathcal{O}$-attraction for the VS system.

Theorem 6.7-3. Let the domain G_0 be $\mathcal{O}$-attracting for the VS system (6.65), then the domain G_δ is attracting for the S_μ system (6.55)-(6.56).

Proof. Consider for $t \geqslant t_0$ an arbitrary V solution $(\boldsymbol{x}(t), \mu(t))$ to the S_μ system (6.55)-(6.56) and assume that the projection of this V solution onto $\mathbb{R}^2_x$ is $\boldsymbol{x}(t) \notin G_\delta$. Then, in view of (6.55), in a time interval at most $2/\alpha$ long, i.e. for $t \geqslant t_0 + 2/\alpha$ the controlled process considered is defined by the solution of equation (6.65).

According to Definition 6.7-3, there is the following alternative for the solutions of equation (6.65): either there exists a finite time instant $\bar{t}$ when $\boldsymbol{x}(\bar{t}) \in G_0$ or else $\lim\limits_{t\to\infty} \| \boldsymbol{x}(t) \| = 0$. Without any loss of generality assume, for definiteness, that $\boldsymbol{x}(t) \in G^+_\delta = \{\boldsymbol{x}: \sigma^+(\boldsymbol{x}) > 0\}$ and examine how the function

$$\sigma^+(\boldsymbol{x}(t)) = \sigma(\boldsymbol{x}(t)) - \delta \mid x(t) \mid \tag{6.66}$$

varies along the solutions of the VS system. If the first part of the alternative is true, then, as can be seen from (6.66), there exists another time instant $t_1 < t'$ such that $\sigma^+(\boldsymbol{x}(t_1)) = 0$. If $\boldsymbol{x}(t)$ is such that $\lim\limits_{t\to\infty} \| \boldsymbol{x}(t) \| = 0$ and for all $t \geqslant t_0$ the function $\sigma^+(t)$ retains its sign, i.e. $\sigma^+(\boldsymbol{x}(t)) > 0$, then in view of $O_x \in G_\delta$ we have for such processes $\lim\limits_{t\to\infty} \boldsymbol{x}(t) \in G_\delta$. This implies that the domain G_δ is attracting, which proves the theorem.

Thus everything has reduced to the conditions that ensure the $\mathcal{O}$-attraction of G_0 for the VS system. Various versions of the respective conditions have been considered in the theory of automatic control systems with variable structure (see, e.g., Refs. 24, 25). To complete the discussion we provide another theorem concerning the sufficient conditions for $\mathcal{O}$-attraction.

Theorem 6.7-4. Let for a constant $c > 0$ and $\xi = 0$ there hold (6.60) and the inequality

$$\inf_{t \geqslant t_0} ca_2(t) > 1 \tag{6.67}$$

Then the domain G_0 is $\mathcal{O}$-attracting for the VS system (6.65).

It is to be noted that the condition (6.67) is sufficient and narrows the class of $\mathcal{P}_x$-processes for which the property of G_δ-attraction is ensured. This inequality will be employed below to simplify certain considerations for this case. Where the inequality (6.67) is no longer valid, one may resort to other conditions, presented, for example in Refs. 24, 25.

Proof. Consider for $t \geqslant t_0$ an arbitrary solution $x(t) = (x(t), \dot{x}(t))$ of equation (6.65) subject to the initial condition $x(t_0) \notin G_0$. Assume for definiteness that $\sigma(x(t_0)) > 0$. With (6.65) it is not hard to see that along this solution the variation of $\sigma(t)$ may be defined as

$$\dot{\sigma} = -\frac{ca_2 - 1}{c}\sigma - \left(\frac{ca_2 - 1}{c} - ca_1\right)x - ck^0 |x|, \quad t \geqslant t_0 \tag{6.68}$$

Analysis of (6.68) indicates that for $t \geqslant t_0$ under the above assumption on the sign of $\sigma(t)$ the derivative $\dot{\sigma}(t) < 0$ provided the conditions of Theorem 6.7-4 are met. Hence, two possibilities exist, namely, there will be found a time constant t' when $\sigma(t') = 0$, or else $\sigma(t) \to 0$ as $t \to \infty$. The first case corresponds to $\mathcal{O}$-attraction. In what follows we wish to demonstrate that if the other alternative is realized then $x(t) \to O_x$ as $t \to \infty$ and, consequently, the domain G_0 is $\mathcal{O}$-attracting. The convergence to zero will not be established directly from (6.68) although such a possibility does exist. Instead we demonstrate this convergence indirectly with the help of the differential constraint (6.30) and a well known statement which is contained in many papers and is quoted below for convenience of further reference.

Lemma 6.7-1. Let for $t \geqslant t_0$ and $c > 0$

$$\dot{x}(t) = -\frac{1}{c}x(t) - \frac{1}{c}\chi(t)$$

and the continuous function $\chi(t) \to 0$ as $t \to \infty$. Then for any $x(t_0)$

$$\lim_{t\to\infty} \dot{x}(t) = \lim_{t\to\infty} x(t) = 0$$

Proof. For $t \geqslant t_0$ and an initial condition $x_0 = x(t_0)$ the solution to the above equation has the form

$$x(t) = x_0 \exp\left(-\frac{t}{c}\right) - \frac{1}{c}\int_{t_0}^{t} \chi(\tau) \exp\left(-\frac{t-\tau}{c}\right) d\tau$$

so that we have for $t \geqslant t_0$

$$| x(t) | \leqslant | x_0 | \exp\left(-\frac{t}{c}\right) + \frac{1}{c} \int_{t_0}^{t} | \chi(\tau) | \exp\left(-\frac{t-\tau}{c}\right) d\tau \quad (6.69)$$

Let ε be an arbitrary, fixed positive number, however small. It is quite obvious that there exists a time instant $t_1(\varepsilon)$ such that for all $t \geqslant t_1(\varepsilon)$ $| x_0 | \exp(-t/c) \leqslant \varepsilon/2$. Let us represent the integral

$$\int_{t_0}^{t} \chi(\tau) \exp(\tau/c)\, d\tau$$

as a sum of two integrals $I_1 + I_2$, where I_1 is taken from t_0 to t_2 and I_2 is taken from t_2 to t, the integrand being the same. Here $t_2 = t_2(\varepsilon)$ is such that for all $t \geqslant t_2$ $| \chi(t) | \leqslant \varepsilon/4$. It is quite clear that such a t_2 does exist and the integral I_1 is bounded by a number $l(\varepsilon)$, i.e., $I_1 \leqslant l(\varepsilon) < \infty$. For I_2 we get the following bound valid for all $t \geqslant t_2(\varepsilon)$:

$$I_2 \leqslant \frac{\varepsilon}{4} c\,(e^{t/c} - e^{t_2/c})$$

Let now $t_3 = t_3(\varepsilon)$ be a time instant such that for all $t \geqslant t_3$

$$\frac{l(\varepsilon)}{c} e^{-t/c} \leqslant \frac{\varepsilon}{4}$$

Considering (6.69) for $t \geqslant \max\{t_1(\varepsilon), t_2(\varepsilon), t_3(\varepsilon)\}$, we obtain that $| x(t) | \leqslant \varepsilon$, and, consequently, $\lim_{t\to\infty} x(t) = 0$. In this case the equation $\dot{x} = -x/c - \gamma/c$ yields $\lim_{t\to\infty} \dot{x}(t) = 0$, thus proving the lemma.

Application of Lemma 6.7-1 for the analysis of those solutions which realize the second possibility ($\sigma(t) \to 0$ as $t \to \infty$) of Theorem 6.7-4 establishes that for the respective controlled processes $\boldsymbol{x}(t) = [x(t), \dot{x}(t)]^{\mathrm{T}}$ there holds $\lim_{t\to\infty} \boldsymbol{x}(t) = O_x \in G_0$. Therefore, the domain G_0 is $\mathcal{O}$-attracting for the VS system. Q.E.D.

By way of comment to Theorems 6.7-3 and 6.7-4 we note that under the conditions of these theorems and the relations between the $\mathcal{P}_x$ parameters and those of the control algorithm, all controlled processes $\boldsymbol{x}(t)$ in the S_μ system considered may be divided into two families, namely those of $\boldsymbol{x}^{\mathrm{I}}(t)$ and $\boldsymbol{x}^{\mathrm{II}}(t)$ processes. For the $\boldsymbol{x}^{\mathrm{II}}$ processes, the above considerations ensure, generally speaking, only the fact that $\lim_{t\to\infty} \| \boldsymbol{x}^{\mathrm{II}}(t) \| = 0$. It is noteworthy that for each $\boldsymbol{x}^{\mathrm{II}}$ process there exists a finite time instant from which $\delta | x^{\mathrm{II}}(t) | < |\sigma(\boldsymbol{x}^{\mathrm{II}}(t))|$. This implies that such controlled processes do not belong to G_δ from some time instant. It is not hard to see that this inequality is helpful in establishing the convergence to zero of $\boldsymbol{x}^{\mathrm{II}}$ processes without recourse to the fact established in Lemma 6.7-1.

For each x^{I} process, a finite time instant t' is ensured when the process reaches G_δ. Under the circumstances either $\mu(t') = -\operatorname{sgn}\sigma(t')$, or prior to t' such a controlled process appears in G_0. It can be readily demonstrated that in fact all x^{I} processes attain G_0 in a finite time, and it is this fact that is required in the conditions of Theorem 6.7-3 to ensure that $x(t) \in G_\delta$. For the said time interval $\Delta t = t_1 - t$ where t_1 is a time instant when for some x^{I} process belonging to G_δ the equality $\sigma(x(t_1)) = 0$ holds for the first time, the estimate from above is the quantity $\delta/c\varkappa$, i.e. for any such controlled process

$$\Delta t \leqslant \delta/c\varkappa \tag{6.70}$$

where

$$\varkappa = k^0 - \bar{\bar{k}}^0 > 0 \tag{6.71}$$

Here the constant $\bar{\bar{k}}^0$ is computed with the system parameters as

$$\bar{\bar{k}}^0 = \max_{|\xi|\leqslant 1} \sup_{t\geqslant t_0} \left| a_1 - a_2\left(\frac{1+\delta\xi}{c}\right) + \left(\frac{1+\delta\xi}{c}\right)^2 \right| \tag{6.72}$$

To prove the validity of the estimate (6.70) consider the "worst" situation with the controlled process at $t = t'$ being at the boundary ∂G_δ of the domain G_δ. For definiteness we assume that this boundary is the domain $G^+(x) = 0$.

Put down for this case the equation describing the variation of the function $\lambda(t) = \sigma(t)/\delta\,|\,x(t)\,|$ for the considered process at $t \geqslant t'$ observing that, by assumption, $\lambda(t') = 1$. After elementary algebra we arrive, similarly to (6.63), at

$$\lambda(t) = 1 + \frac{c}{\delta}\int_{t'}^{t} [(a_1 - a_2\zeta + \zeta^2)\operatorname{sgn} x - k^0]\,d\tau \tag{6.73}$$

where $\zeta = (1 + \delta\xi)/c$ and $\xi = \xi(t) = \lambda(t)\operatorname{sgn} x(t)$. Given (6.71) is met, the integrand in (6.73) is negative, therefore, using the expression for $\varkappa$ from (6.71) we arrive at the following bound:
$\lambda(t) \leqslant 1 - (t - t')\,c\varkappa/\delta$
Observing that for $\sigma = 0$ by definition $\lambda(x) = 0$, we readily obtain from this inequality the desired estimate (6.70).

The foregoing considerations suggest the following assertion concerning the properties of controlled processes in the S_μ system (6.55)-(6.56).

Theorem 6.7-5. Let for the S_μ system (6.55)-(6.56) at constant $c > 0$ the relationships (6.59), (6.61), (6.67) and (6.71) be met. Then all controlled processes $x(t)$ in such a system divide themselves in two types of processes, $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. Beginning from a certain time $x^{\mathrm{I}}(t) \in G_\delta$ the estimate (6.58) holds for them also. Likewise, from some time instant $x^{\mathrm{II}}(t) \notin G_\delta$ and $\lim_{t\to\infty} \| x^{\mathrm{II}}(t) \| = 0$.

This theorem pinpoints the sufficient conditions that ensure the convergence to O_x of all controlled processes in the S_μ system witk integral COFB at any values of $\mathcal{P}_x$ parameters and any laws of their variation within (4.2). In accord with the earlier introduced terminology this implies that the $\mathcal{P}_x$-process is $\mathcal{A}_\mu$ controlled with the control algorithm (6.53)-(6.55) provided (6.67) is satisfied.

The theorem also brings forth those controlled processes whose dynamics is close to the specified performance. We shall evaluate this topic in more detail in Problem F below. At the moment we demonstrate that the $\mathcal{P}_x$-process is steady $\mathcal{A}_\mu$ controlled under the conditions stated in Theorem 6.7-5.

Problem E. According to the aforestated definition, the $\mathcal{P}_x$-process is steady $\mathcal{A}_\mu$ controlled if for all controlled processes in the S_μ system (6.55)-(6.56) the bound (4.4a) is valid. We shall formulate this result in the form of a lemma yielding an expression for the constant N in the bound (4.4a).

Lemma 6.7-2. If all the conditions of Theorem 6.7-5 are met, then for every controlled process of the S_μ system (6.55)-(6.56) for $t \geqslant t_0$

$$\| x(t) \| \leqslant \frac{c_m}{\delta}\left(\frac{1+\delta+c}{c}\right)^2 \exp(2P/\alpha) \| x(t_0) \| \tag{6.74}$$

where

$$P = \max_{|\xi| \leqslant 1} \left[\sup_{t \geqslant t_0} | a_1(t) + \xi k^0 |, \sup_{t \geqslant t_0} (1 + | a_2(t) |)\right]$$

It is worth noting that the bound (6.74) coincides with (4.4a) for

$$N = \frac{c_m}{\delta}\left(\frac{1+\delta+c}{c}\right)^2 \exp(2P/\alpha)$$

Proof. Consider for $t \geqslant t_0$ an arbitrary V solution to the S_μ system for which $\| x(t_0) \| = r$, where r is some positive number. Let us estimate the variation of $\| x(t) \|$ on the interval $[t_0, t_0 + 2/\alpha]$. Introducing the state coordinates $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$ carries equation (6.56) to the form

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -a_1(t) x_1(t) - a_2(t) x_2(t) - k^0 \mu(t) | x_1(t) |, \quad t \geqslant t_0 \tag{6.75}$$

or in a vector-matrix form

$$\dot{x}(t) = A(t) x(t) - b(t) k(t) \mu(t) x(t), \quad t \geqslant t_0$$

where

$$A(t)=\begin{bmatrix} 0 & 1 \\ -a_1(t) & -a_2(t) \end{bmatrix}, \quad b(t)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $k(t) = [k^0 \operatorname{sgn} x_1(t), 0]$.

We rewrite this equation in a more compact form

$$\dot{x}(t) = A_\mu(t)\, x(t), \quad t \geqslant t_0 \tag{6.76}$$

where $A_\mu(t) = A(t) - b(t)\, k(t)\, \mu(t)$.

Equation (6.76) is equivalent to the integral equation

$$x(t) = \int_{t_0}^{t} A_\mu(\tau)\, x(\tau)\, d\tau + x(t_0), \quad t \geqslant t_0$$

Assuming $P = \sup\limits_{t \geqslant t_0} \| A_\mu(t) \|$—here the matrix norm is induced by the vector norm being considered—and conducting norm-estimation in the integral equation we get

$$\| x(t) \| \leqslant \| x(t_0) \| + P \int_{t_0}^{t} \| x(\tau) \|\, d\tau, \quad t \geqslant t_0$$

Invoking the Gronwall-Bellman inequality and substituting the value $t \leqslant t' = t_0 + 2/\alpha$ leads us to the estimate

$$\| \dot{x}(t') \| \leqslant \| x(t_0) \|\, e^{P(t-t_0)} \leqslant \| x(t_0) \|\, e^{2P/\alpha} = r e^{2P/\alpha} \tag{6.77}$$

which is valid for any controlled process $x(t)$ for all $t \in [t_0, t']$.

For the point $x(t')$ two positions are possible in $\mathbb{R}^2_x$, namely, (1) $x(t') \in G_\delta$, and (2) $x(t') \notin G_\delta$. In case (1) by Theorem 6.7-5 $x(t) \in G_\delta$ for all $t \geqslant t'$ so that the estimate (6.58) is valid and hence $\| x(t) \| \leqslant \| x(t') \| (1 + c + \delta)/c$ for $t \geqslant t'$. Making use of (6.77) in this inequality yields for such controlled processes the bound

$$\| x(t) \| \leqslant \frac{1+c+\delta}{c}\, r \exp(2P/\alpha), \quad t \geqslant t_0 \tag{6.78}$$

In case (2), $\mu(t') = -\operatorname{sgn} \sigma(t')$ and under the conditions stated in Lemma 6.7-2 the function $| \sigma(t) |$ vanishes along the considered V solution, i.e., $| \sigma(t) | \leqslant | \sigma(t') |$ for $t \geqslant t'$. The controlled process under study $x(t)$ for $t \geqslant t'$ can belong either to the first type (x^{I} process) or to the second type (x^{II} process). If it belongs to the second type, then $x^{\mathrm{II}}(t) \notin G_\delta$ and consequently for all $t \geqslant t'$ $\delta | x^{\mathrm{II}}(t) | < | \sigma(t) |$. Then the use of the differential constraint $\dot{x} = -x/c - \sigma(t)/c$ leads for $t \geqslant t'$ to the following sequence of inequalities:

$$\| x^{\mathrm{II}}(t) \| = | \dot{x}^{\mathrm{II}}(t) | + | x^{\mathrm{II}}(t) | \leqslant \frac{1+c}{c} | x(t) | + \frac{1}{c} | \delta(t) |$$
$$\leqslant | \delta(t) | (1+\delta+c)/c$$

We strengthen this inequality with account of (6.77), the fact that $|\sigma(t)| \leqslant |\sigma(t')|$ for $t \geqslant t'$, and that always $|\sigma(t')| \leqslant c_m \|x(t')\|$, where $c_m = \max(1, c)$. This results in the bound

$$\|x^{II}(t)\| \leqslant r c_m \frac{1+\delta+c}{c} \exp(2P/\alpha), \quad t \geqslant t_0 \tag{6.79}$$

Now if in case (2) the controlled process belongs to the $x^I_{\vec{r}}$ family, it takes a finite time to reach the boundary of G_δ, i.e., $x^I(t'') \in \partial G_\delta$ at some t''. Since $x^{II}(t) \notin G_\delta$ for t in $[t', t'']$, then $\delta |x^I(t)| < |\sigma(t)|$ on this interval and, therefore, the bound $\|x^I(t)\| \leqslant \|x(t')\| c_m (1 + \delta + c)/c$ proved above is valid for all t in $[t', t'']$. According to Theorem 6.7-5 the process under investigation $x^I(t) \in G_\delta$ for $t \geqslant t''$, so that for this process the inequality (6.58) is valid for $t_1 = t''$ and $t > t''$. The account of this fact and the inequality established above leads us to conclude that for all $t \geqslant t'$

$$\|x^I(t)\| \leqslant \frac{c_m}{\delta} \left(\frac{1+\delta+c}{c}\right)^2 \|x(t')\|$$

We conclude the derivation of the bound for the x^I processes considered for $t \geqslant t_0$ by observing (6.77) in the expression just obtained to get

$$\|x^I(t)\| \leqslant r \frac{c_m}{\delta} \left(\frac{1+\delta+c}{c}\right)^2 \exp(2P/\alpha), \quad t \geqslant t_0 \tag{6.80}$$

From the bounds (6.78)-(6.80) the bound immediately follows

$$\|x(t)\| \leqslant r \frac{c_m}{\delta} \left(\frac{1+\delta+c}{c}\right)^2 \exp(2P/\alpha), \quad t \geqslant t_0 \tag{6.81}$$

that is valid for every controlled process of the S_μ system being considered.

Now we note that at any t, for the vector norm adopted in this work,

$$\|A_\mu(t)\| = \max\left[\max_{|\xi| \leqslant 1} |a_1(t) + \xi k^0|, \; 1 + |a_2(t)|\right]$$

therefore, the number P may be defined as

$$P = \max_{|\xi| \leqslant 1} \left[\sup_{t \geqslant t_0} |a_1(t) + \xi k^0|, \; \sup_{t \geqslant t_0} (1 + |a_2(t)|)\right]$$

This concludes the proof of Lemma 6.7-2.

To summarize, the conducted evaluation of solutions to the S_μ system (6.55)-(6.56) suggests that given the conditions of Theorem 6.7-5 are met the $\mathcal{F}_x$-process is steady $\mathcal{A}_\mu$ controlled and for the x^I processes only the exponential bound (6.58) can be warranted

beginning from a certain time instant. Therefore, for all controlled processes in an S_μ system with integral COFB subject to (6.67)

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\mu}\left\{\mathcal{F}_c,\ I_2;\ \frac{c_m}{\delta}\left(\frac{1+\delta+c}{c}\right)^2 \exp(2P/\alpha)\right\} \tag{6.82}$$

Comparison of this expression with (6.51) indicates that the properties of the S_μ system (6.55)-(6.56) differ markedly from those required. The difference between (6.51) and (6.82) suggests that at finite values of system structural parameters the earlier formulated problem of attaining asymptotic insensitivity of system dynamics with respect to parametric perturbations cannot be solved by employing an integral control law in the COFB loop.

Confining ourselves to the consideration of the first-type processes, we may speak of weakly parametrically dependent dynamics achieved in such an S_μ system, or parametric "dissipatism" in accord with the definition given in Section 6.6. To illustrate, the ideal motion is characterized by the equation $c\dot{x}(t) + x(t) = 0$ and the point $\pi_0 = (1, c)$ in the parameter space π characterizes this ideal motion. In the S_μ system (6.55)-(6.56), the motion over the phase space for $x(t) \in G_\delta$ is described by the equation $c\dot{x}(t) + x(t) = -\delta\lambda(t)\, x(t)$, where $\lambda(t)$ is a function vanishing as $t \to \infty$ and such that $|\lambda(t)| \leqslant 1$. To the last equation at every t there corresponds a point $\pi_\lambda = (1 + \delta\lambda(t), c)$ in the π space. The distance between points π_0 and π_λ in terms of the adopted norm is given as

$$\Delta_\pi = \| \pi_0 - \pi_\lambda \| \leqslant \delta \tag{6.83}$$

Since ins general $\lambda(t)$ does not go to zero as $t \to \infty$, the function $\pi_\lambda(t)$ fail to converge to the point π_0 and what can be ensured is only (6.83). It would be natural to interpret this inequality as the presence in S_μ system dynamics of a weak parametric dependence, "dissipatism", characterized by a quantity never exceeding the constant δ. Therefore, the properties of the controlled processes in the S_μ system (6.55)-(6.56) differ from the required performance.

A quantitative insight into the behavior of controlled processes in the S_μ system concerned can be obtained from Fig. 6.15 which depicts the projections of V solutions onto $\mathbb{R}^2_x$ for some fixed values of $\mathcal{P}_x$ parameters a_1 and a_2 from (4.2).

Problem F. Here we are going to consider in more detail the dependence of the dynamic properties of controlled processes upon the $\mathcal{P}_x$ parameters and the control algorithm parameters. To begin with, we note that x^{II} processes depend considerably on the $\mathcal{P}_x$ parameters $a_1(t)$ and $a_2(t)$ and, in general, it is a profound task to draw any conclusions on their properties except for $\lim_{t\to\infty} \| x^{II}(t) \| = 0$. For x^I processes, the effect of parametric disturbances in $a_1(t)$ and $a_2(t)$ on their properties from some time instant is characterized

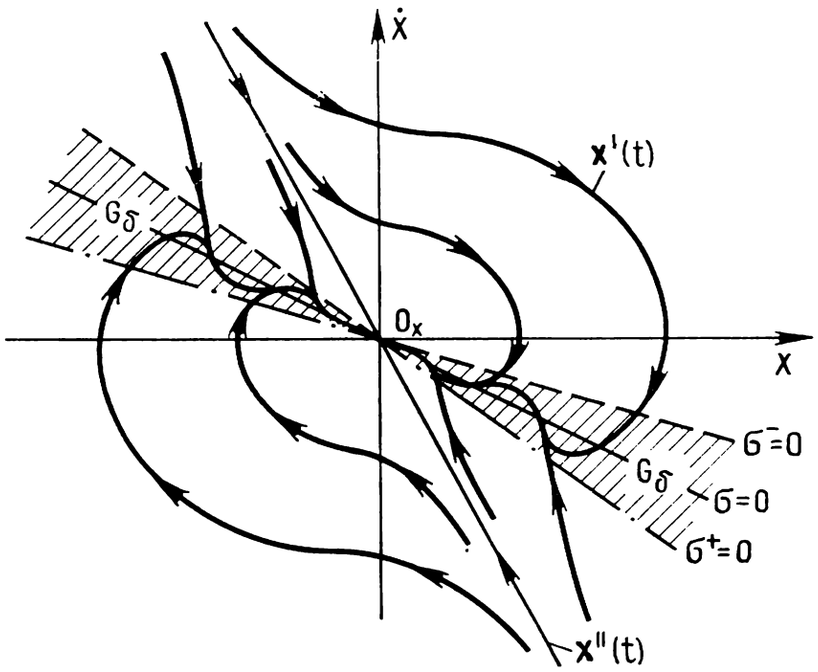

Figure 6.15

by inequality (6.83) and with a sufficiently small structural parameter δ the dynamic behavior of these processes can be made arbitrarily weakly dependent on $a_1(t)$ and $a_2(t)$ and close to the desired dynamic performance. Therefore, in the S_μ system considered the asymptotic independence with respect to bounded parametric disturbances can be achieved only for x^{I} processes by letting δ go to zero. When δ tends to zero, however, the coefficient α in the COFB loop grows to infinity as suggested by (6.61). This implies that in the limit of $\alpha = \infty$ the $\mu(t)$ generation law becomes a bang-bang law. As a result, the S_μ system with an integral COFB law becomes an S_μ system with a bang-bang COFB law, or a VS system. The properties of such systems have been discussed in Section 6.4 where it has been noted that their behavior does not meet the performance specifications because the control action does not belong to the class of continuous time functions. Consequently, the parameter δ should be selected other than zero and such that would ensure the specified approximation of the controlled process behavior to the required performance.

To conclude the topic we recognize that integral COFB is not a means to secure asymptotically accurate control under jumpwise variation of controlled-process parameters. This feedback is capable of providing only a low sensitivity to parametric disturbances, whose magnitude can be selected at will. From this aspect it seems worthwhile to seek for such means and control laws which would warrant asymptotic insensitivity to parameters in the class of controls in question. We shall discuss such control algorithms in Chapter 8.

6.8 Inertial Coordinate-Operator Feedback

In the foregoing sections we demonstrated that integral COFB is generally incapable of providing a closed-loop control system with dynamics asymptotically independent of parametric disturbances. The discussion of methods warranting this result is reserved until later chapters. This section will be devoted to the evaluation of COFB laws aimed at improving certain properties of binary control systems. These control laws attract some interest as they enable the properties of a closed-loop control system to be driven arbitrarily close to the specified system performance.

Binary control systems with integral COFB (6.55) may experience a sliding mode on the manifold $|\mu| = 1$ where the right-hand side of equation (6.55), defining the differential law of $\mu(t)$ variation, suffers discontinuities. Such sliding modes occur in the loops other than the primary (CFB) loop, hence the constraints imposed on their characteristics are less severe than those imposed on sliding modes in the primary loop.

Notwithstanding this advantage, there exist a number of reasons which make the occurrence of sliding modes in other loops fairly disadvantageous. One of the reasons is due to the lack of suitable numerical integration techniques for the derivative systems with sliding modes. Therefore, the implementation of control algorithms such as (6.54) and (6.55) with the on-off element technology may face certain difficulties. It would be interesting in this connection to evaluate also the COFB laws that retain the above-established properties for the S_μ systems (6.55)-(6.56) while no sliding modes occur.

A first of such laws we are going to consider is one with inertial coordinate-operator feedback. The generation of the operator variable $\mu(t)$ is governed by the equation

$$\dot{\mu}(t) = -\alpha \{\mu(t) + \operatorname{sgn} \sigma(x(t))\}, \quad t \geqslant t_0 \tag{6.84}$$

where, as before, $\sigma(x) = -cx$, $c = (1, c)$ and α is a positive constant representing a structural parameter of the algorithm.

The closed-loop control system with such COFB, denoted as before by S_μ, is given by equation (6.56) and (6.84). The block diagram representing an S_μ system with inertial COFB in which the function $k(\mu(t))$ is defined by relationships (6.53)-(6.54) is depicted in Fig. 6.16.

The solutions to the S_μ system (6.56), (6.84) occur as pairs of functions $(x(t), \mu(t))$. These solutions are understood in the same sense as in Section 6.7. It is an easy matter to verify that any solution $(x(t), \mu(t))$ generated by arbitrary initial conditions (x_0, μ_0) exists for all $t \geqslant t_0$.

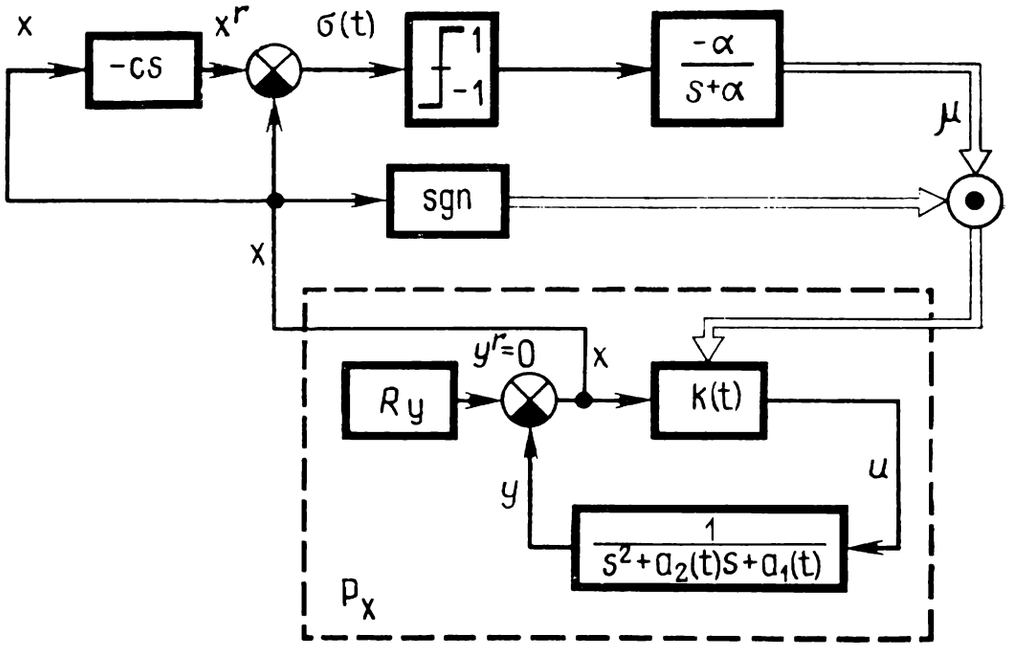

Figure 6.16

Amidst all solutions of the S_μ system (6.56), (6. 84) we single out the V solutions, corresponding to the initial conditions from the set $V = \{(x, \mu): x \in \mathbb{R}^2, |\mu| \leqslant 1\}$, and the V' solutions generated by the initial conditions from the set $V' = \{(x, \mu): x \in \mathbb{R}^2, |\mu| < 1\}$. Analysis of equation (6.84) indicates that any V solution (V' solution) remains such for all $t > t_0$. Since in this case $|\mu(t)| \leqslant 1$ for $t \geqslant t_0$, then the condition $|k(\mu)| \leqslant k^0$ holds, and hence the required limitation of the CFB loop gain takes place.

Our further investigation of the properties of the S_μ system (6.56), (6.84) will be confined to the study of V' solutions only. This restriction is due to the fact that under the conditions stated below any solution $(x(t), \mu(t))$ of such an S_μ system appears, in a finite time, in the class of V' solutions, or else for such solution $\lim_{t\to\infty} \|x(t)\| = 0$. We note also that Definitions 6.7-1 and 6.7-2 will be used in this section along with the term V solutions being replaced with V' solutions in their context.

We begin our investigation into the properties of solutions to the S_μ system with inertial COFB by noting that for its V' solutions the exact fulfillment of $\sigma(x(t)) \equiv 0$, generally speaking, cannot be accurately kept at a finite value of the parameter α, in much the same way as is the case with the S_μ system with integral COFB investigated in the preceding section. This assertion can be easily verified by analysis—this will be valid for the S_μ system concerned as well—of (6.57) for $\dot{\sigma}(x(t))$ computed for some V' solution. Therefore, for S_μ systems with inertial COFB we may speak only of approximate fulfillment of the above equality. In what follows this approximation will be understood, as before, in the sense of $x(t) \in G_\delta$

beginning with some time instant for a V' solution $(x(t), \mu(t))$ of the S_μ system (6.56), (6.84) at a sufficiently small value of the parameter δ, where G_δ is defined by relationships (6.41). With such an approach, the problem of qualitative investigation into the S_μ system behavior reduces to solving the set of Problems A-F formulated in the previous section.

First of all we are bound to note that the solution of Problem A, devoted to the asymptotic behavior of the phase-plane projections of those solutions for which $x(t) \in G_\delta$, is not connected with the type of control algorithm being employed. Therefore, for the S_μ system (6.56), (6.84) this problem is also solved by way of Theorem 6.7-1 and Corollary 6.7-1, and, of course, in this case also the estimate (6.58) is valid for the norms of controlled processes under investigation.

Problem B. Some specific features of the S_μ system being analyzed manifest themselves in solving Problem B focused on the evaluation of the conditions ensuring $x(t) \in G_\delta$ for some V' solutions. Therefore, we display the solution of this problem in depth. The principal result for this case is contained in the following statement.

Theorem 6.8-1. Let h be a number between zero and unity and for the S_μ system (6.56), (6.84)

$$k^0 > \bar{k}^0/(1-h) \tag{6.85}$$

where

$$\bar{k}^0 = \max_{|\xi|=1} \sup_{t \geqslant t_0} |a_1(t) - a_2(t)\zeta + \zeta^2| \tag{6.86}$$

$$\alpha \geqslant \frac{c}{\delta}(k^0 + \bar{\bar{k}}^0) \ln \frac{2}{h} \tag{6.87}$$

where $\zeta = (1 + \delta\xi)/c$, and $\bar{\bar{k}}^0$ is defined by (6.72). Then the domain G_δ is G_0 invariant for such an S_μ system.

It will be recalled that G_0 is the set of points satisfying $\sigma(x) = 0$. Theorem 6.8-1 asserts that given (6.85) and (6.87) any V' solution that takes off from the line $\sigma(x) = 0$ will not leave its neighborhood, the domain G_δ, at all subsequent t.

The constant h appearing in the conditions of Theorem 6.8-1 may be selected at will. It is remarkable, however, that as this constant increases so does the CFB loop gain while the COFB loop gain decreases. This implies that the choice of an h value is a matter of tradeoff.

Proof. Let $(x(t), \mu(t))$ be a V' solution meeting the conditions of the theorem, i.e., such that satisfies $\sigma(x(t_1)) = 0$ at some time instant $t_1 > t_0$. We investigate this V' solution for $t \geqslant t_1$. Similar to the proof of Theorem 6.7-2 we convince ourselves that if this V' solution for $t \in [t_1, t_1 + \Delta]$ (here Δ is a positive number) has the

form $(O_x, \mu(t))$, where $\mu(t)$ is an absolutely continuous function for $t \geqslant t_1$ such that $\| \mu(t) \| < 1$ for these t, and $\| \dot{\mu}(t) \| < 2\alpha$ almost everywhere on $[t_1, \infty)$, then this form will be retained for all $t \geqslant t_1$. Because the point $O_x \in G_\delta$, the domain G_δ is G_0 invariant.

Assume now that for the V' solution in question the controlled process $x(t_1) \neq O_x$, but then it is clear that for all $t \geqslant t_1$ $x(t) \neq O_x$. Suppose that for the said V' solution there exists a time instant $t' > t_1$ when the controlled process first reaches the boundary of G_δ, i.e., $x(t') \in \partial G_\delta$, where ∂G_δ is defined by (6.62). It is quite obvious then that there exists an interval $(t_1', t']$ $t_1' \geqslant t_1$ such that $\sigma(x(t_1')) = 0$ and the function $\sigma(x(t))$ retains its sign for $t \in (t_1', t']$. In what follows we assume for definiteness that this is a plus sign.

We wish to demonstrate that given (6.87), then $\mu(t') \leqslant -1 + h$. Assume the contrary, that is, $\mu(t') > -1 + h$ at this time instant. Like in the proof of Theorem 6.7-2, we introduce the function $\lambda(t) = \sigma(x(t))/\delta \mid x(t) \mid$, defined for the said V' solution for all $t \geqslant t_1'$, and evaluate its variation on the time interval from t_1' when $\lambda(t_1') = 0$ to t' when $\lambda(t') = 1$. Recall that the function $\lambda(t)$ characterizes the slope of the broken line tracing out the position of the state point $x \in G_\delta$. This function is defined by the integral equation (6.63) rewritten here as

$$\lambda(t) = \frac{c}{\delta} \int_{t_1'}^{t} [(a_1 - a_2\zeta + \zeta^2) \operatorname{sgn} x + k^0\mu] \, d\tau \tag{6.88}$$

where $\zeta = (1 + \delta\xi)/c$ and $\xi = \lambda(t) \operatorname{sgn} x(t)$.

Now we note that for all t in $[t_1', t']$ according to (6.84) and the assumption made above the variable $\mu(t)$ is the solution of the equation $\dot{\mu} = -\alpha(\mu + 1)$ and therefore for $t \in [t_1', t']$ its variation law may be given as $\mu(t) = -1 + [\mu(t_1') + 1] \exp[-\alpha(t - t_1')]$.

Since by assumption $\mu(t') > -1 + h$, then from the last equality we get the following obvious bound for the length of the $t' - t_1'$ interval, viz.

$$t' - t_1' < \frac{1}{\alpha} \ln \frac{2}{h} \tag{6.89}$$

We estimate the absolute value of the integrand in (6.88) using the designation introduced in (6.72) together with inequalities (6.87) and (6.89). Omitting the intermediate manipulations we get as a final result at $t = t'$

$$\lambda(t) \leqslant \frac{c}{\delta} (\bar{k}^0 + k^0)(t' - t_1') \leqslant \frac{\alpha}{\ln(2/h)} (t' - t_1') < 1$$

The last inequality implies that $\lambda(t') < 1$, which contradicts to the initial assumption of $x(t') \in \partial G_\delta$, when $\lambda(t')$ should have been equal to unity. Thus we have proved (here we include into the pat-

tern the second case with $\sigma(x(t)) < 0$ for t in $(t_1', t']$ for which the proof is analogous) that at t' $|\mu(t')| \geqslant 1 - h$.

Let now $x(t')$ be the aforementioned point where $\sigma^+(x(t')) = 0$. Consider the behavior of the controlled process $x(t)$ for $t \geqslant t'$ and under the stated initial condition. It is quite obvious that there exists a time interval $[t', t'']$ on which $\sigma(x(t)) > 0$. Then for all t in $[t', t'']$ by far $\mu(t) \leqslant -1 + h$ as can be seen from equation (6.84). Put down the expression for the derivative $\dot{\sigma}^+(t) = \dot{\sigma}^+(x(t))$ on the considered V solution taken subject to $\sigma^+(x(t)) = 0$. It is an easy matter to see that this equation coincides with (6.64). From analysis of (6.64) we have, after account of all the remarks made above and the proved inequality $\mu(t) \leqslant -1 + h$ valid for all $t \in [t', t'']$:

$$\dot{\sigma}^+(t') \,|_{\sigma^+(t')=0} \leqslant c\,[\overline{k}^0 - (1-h)\,k^0]\,|\,x(t')\,|$$

Having incorporated into this bound the inequality (6.85) we can verify that the considered process $x(t)$ for $t > t'$ cannot leave into the area G_δ^+ from the boundary $G_\delta^+(x) = 0$ of G_δ. Further argument coincides with the concluding considerations made in proving Theorem 6.7-2. Applying this argument we see that the domain G_δ is G_0 invariant for the S_μ system (6.56), (6.84). Q.E.D.

Comparing the relationships suggested by Theorem 6.7-2 and 6.8-1 for the structural parameters k^0 and α involved in the control algorithm we see that the larger values always are due to the S_μ system with inertial COFB. Conditionally, we may treat this growth of the coefficients as a cost we have to pay for going over from the COFB law (6.55) to the simpler in implementing and exhibiting better dynamics COFB law (6.84).

Theorem 6.8-1 defines the conditions under which any controlled process in the S_μ system (6.56), (6.84) which takes off from the line $\sigma(x) = 0$ subsequently obeys the bound (6.55). If inequality (6.59) is also valid then in the limit we have $\lim_{t\to\infty} \|x(t)\| = 0$, i.e., such controlled processes converge to O_x. It is worth noticing that having chosen the structural parameters of control system, the estimate (6.58) may be refined to point out better values for the constants involved in the bound. Using elements of the argument employed in proving Theorem 6.8-1 we can prove the validity of the following statement.

Remark 6.8-1. Let at some fixed h taken between zero and unity the parameters k^0 and α of the control algorithm (6.54), (6.84) be selected in accord with (6.85), (6.87) and the condition $(1 - h)\,k^0 > \overline{\overline{k}}^0$. Let also $\overline{\overline{h}} = 1 - \overline{\overline{k}}^0/k^0$; obviously, $h < \overline{\overline{h}}$. Then for the controlled processes $x(t)$ considered in Theorem 6.8-1 for all $t \geqslant t_1$ we have

the bound similar to (6.58)

$$\| x(t) \| \leqslant \frac{1+c+\delta_h}{c} \| x(t_1) \| \exp\left[-\frac{1-\delta_h}{c} (t - t_1) \right] \tag{6.90}$$

where $\delta_h = \delta\varphi(h)$, and $\varphi(h) = \ln(2/\bar{\bar{h}})/\ln(2/h) < 1$.

This bound implies that in fact $x(t) \in G\delta_h \in G_\delta$.

Now we go over to investigating the convergence of controlled processes in the S_μ system (6.56), (6.84) toward the domain G_δ and the asymptotic behavior of those of them for which $x(t) \notin G_\delta$ beginning from a certain time instant. In accord with our terminology, such topics constitute the subject matter of Problems C and D.

Problems C and D. For S_μ systems with inertial COFB the scheme of argument employed in Section 6.7, while investigating similar topics for S_μ systems with integral COFB, is not applicable directly. One of the reasons for this is that at no $t \geqslant t_0$ solutions of the VS system (6.65) can define the phase-space projections of the considered V solutions of the S_μ system (6.56), (6.84). The statements of Section 6.7 may only be used to analyze the behavior of those V solutions for which $|\mu(t_0)| = 1$. The evaluation of the properties of V' solutions may be carried out in the framework of similar argument, but after appropriate alterations of certain concepts. We display these alterations below.

Let h be a number taken in between zero and unity. We shall refer to a dynamic system described by the differential equation

$$\ddot{x}(t) + a_2(t)\dot{x}(t) + a_1(t)x(t) = k_h^0(t) |x(t)| \operatorname{sgn}\sigma(t) \tag{6.91}$$

for $t \geqslant t_1$, where $k_h^0(t)$ is a continuous time function such that for all $t \geqslant t_1$

$$k_h^0(t) > k^0(1-h) \tag{6.92}$$

as a *VS_h system.*

The connection between the VS_h system (6.91)-(6.92) and the S_μ system (6.56), (6.84) will be established below. If for some V' solution of S_μ system on a certain time interval $[t_1, t_2]$ such that $t_2 - t_1 > (1/\alpha)\ln(2/h)$ the function $\sigma(x(t))$ does not experience a sign reversal, then for $t \in [t_1', t_2]$, where $t_1' < t_2$ is a time instant pinpointed below the projection of V' solution on the phase space is determined by the VS system solution at some function $k_h^0(t)$. To demonstrate, under the above assumption $t_2 - t_1 > (1/\alpha)\ln(2/h)$ and with equation (6.84) describing the variation law for $\mu(t)$ we have for t in $[t_1, t_2]$

$$\mu(t) = -\operatorname{sgn}\sigma(t) + 1(\operatorname{sgn}\sigma(t) + \mu(t_1))\exp[-\alpha(t - t_1)]$$

Because for V' solutions $|\mu(t)| < 1$, this expression for the same t is equivalent to

$$-\mu(t)\operatorname{sgn}\sigma(t) = 1 - [1 + \mu(t_1)\operatorname{sgn}\sigma(t)]\exp[-\alpha(t - t_1)]$$

If now t_1' is such that $t_1' - t_1 = (1/\alpha) \ln (2/h)$ then it is an easy matter to see that for all $t \geqslant t_1'$

$$-\mu(t) \operatorname{sgn} \sigma(t) \geqslant 1 - h$$

Letting $k_h^0(t) = -k_\mu^0(t) \operatorname{sgn} \sigma(t)$ carries equation (6.56) to coincide with (6.91), and validizes for $k_h^0(t)$ inequality (6.92) for $t \geqslant t_1'$. This proves that the behavior of controlled processes in the S_μ system (6.56), (6.84) for t in $[t_1', t_2]$ can be determined by the solutions of the VS_h system (6.91)-(6.92).

The concept of VS_h system allows us to exploit the argument from the previous section. We shall assume that for the VS_h system Definition 6.7-3 (h) takes place of Definition 6.7-3 where the term VS system gives way to the term VS_h system. With this nomenclature it would be a rather easy matter to prove the theorem on the convergence of the projections of V' solutions of the S_μ system (6.56), (6.84) to the domain G_δ. For binary control systems with inertial COFB this statement is the analog of Theorem 6.7-3 for the systems with integral COFB.

Theorem 6.8-2. Let the domain G_0 be $\mathcal{O}$-attracting for the VS_h system (6.91)-(6.92). Then G_δ is attracting for the S_μ system (6.56), (6.84).

We omit the proof of this theorem as it parallels that for Theorem 6.7-3. Expressed another way, this theorem reduces the question of convergence of S_μ system processes to G_δ to the problem of $\mathcal{O}$-attraction [Definition 6.7-3 (h)] for the VS_h system, the latter being a simpler task.

Theorem 6.8-3. Let c and h, $0 < h < 1$, be some positive numbers, and the condition (6.67) be valid and

$$k^0 (1 - h) > \tilde{k}^0 \tag{6.93}$$

where

$$\tilde{k}^0 = \sup_{t \geqslant t_0} | a_1(t) - a_2(t)/c + 1/c^2 | \tag{6.94}$$

Then the domain G_0 is $\mathcal{O}$-attracting for the VS_h system (6.91)-(6.92).

We omit the proof for this theorem as well, as it is based on the analysis of expression for $\dot{\sigma}(x(t))$ derived for the VS_h system and can be reconstructed with the argument used in proving Theorem 6.7-4.

The results of Theorems 6.8-2 and 6.8-3 may be interpreted as follows. Given the conditions of Theorem 6.8-3 are met, for any h fixed between zero and unity all controlled processes $x(t)$ in the S_μ system with inertial COFB are bound to divide themselves in time into two families, viz. x^{I} and x^{II} processes. For x^{II} processes in accord with the property of $\mathcal{O}$-attraction, $\lim_{t \to \infty} \| x^{\mathrm{II}}(t) \| = 0$. For each x^{I} process there exists a finite time instant when it attains G_δ.

For such controlled processes we may also warrant convergence to the origin of the phase space if, once having appeared in G_δ, they never leave it subsequently. According to Theorem 6.8-1 this will be true if, given relationships (6.85) and (6.87) are valid, such controlled processes cross the domain G_0. We demonstrate that under the said conditions this does take place.

Assume for some x^{I} process t' is a time instant when $x^{\mathrm{I}}(t') \in G_\delta$ and for the respective V' solution $(x^{\mathrm{I}}(t), \mu(t)) \mid \mu(t') \mid \geqslant 1 - h$. The existence of such a time instant t' follows from Theorem 6.8-3. Assume that at t posterior and close to t' the function $\sigma(x^{\mathrm{I}}(t))$ experiences no sign reversal as otherwise $\sigma(x(t)) = 0$ and the proof is over. The variation of function $\lambda(t) = \sigma(x^{\mathrm{I}}(t))/\delta \mid x^{\mathrm{I}}(t) \mid$ on the considered V' solution for $t \geqslant t'$ is described by the equation

$$\lambda(t) = \lambda(t') + \frac{c}{\delta}\int_{t'}^{t} [(a_1 - a_2\zeta + \zeta^2) \times \operatorname{sgn} x + k^0\mu]\, d\tau \tag{6.95}$$

where $\zeta = (1 + \delta\xi)/c$, and $\xi = \lambda(t) \operatorname{sgn} x(t)$.

In estimating the absolute value of the right-hand side of (6.95) we invoke (6.72), the inequality $\mid \lambda(t') \mid \leqslant 1$ valid at t', and the fact that owing to the above assumption $\mu(t) \operatorname{sgn} \sigma(t) \leqslant -(1 - h)$ for t close to t' though beyond this point. As a result, from (6.95) we get for $t \geqslant t'$

$$\mid \lambda(t) \mid \leqslant 1 - (c/\delta)\,[k^0(1 - h) - \overline{\overline{k}}^0]\,(t - t')$$

Given

$$k^0(1 - h) - \overline{\overline{k}}^0 > 0 \tag{6.96}$$

the above inequality readily suggests that in the interval $[t', t' + \Delta t]$, where Δt is given by the bound (6.70), the controlled process $x^{\mathrm{I}}(t)$ attains G_0 with a warranty. Given the conditions of Theorem 6.8-1 are met, in subsequent movement $x^{\mathrm{I}}(t) \in G_\delta$.

Theorem 6.8-4. Let the relationships (6.59), (6.67), (6.87), and (6.96) be valid for the S_μ system (6.56), (6.84) at some positive constants c and h, $0 < h < 1$. Then all the controlled processes $x(t)$ in such S_μ system possess the property $\lim_{t\to\infty} \| x(t) \| = 0$ and divide themselves into two families, $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. Beginning from a certain time, for x^{I} processes the bound (6.90) is valid.

Essentially, Theorem 6.8-4 establishes the sufficient conditions under which the $\mathcal{P}_x$-process is $\mathcal{A}_\mu$ controlled. Now we wish to demonstrate that under the conditions concerned the $\mathcal{P}_x$-process is steady $\mathcal{A}_\mu$ controlled. The substantiation of this fact and the derivation of pertinent bounds make up the subject of Problem E.

Problem E. The solution to this problem is given by the ensuing lemma. Its proof is not given as it can be readily developed on the lines of argument used in proving Lemma 6.7-2 and Theorem 6.8-1.

Lemma 6.8-1. Given all the conditions of Theorem 6.8-4 are satisfied, for any controlled process $x(t)$ of the S_μ system (6.56), (6.84), for all $t \geqslant t_0$

$$\| x(t) \| \leqslant \frac{c_m}{\delta} \left(\frac{1+\delta+c}{c} \right)^2 \exp \left[\frac{P}{\alpha} \ln \frac{2}{h} \right] \| x(t_0) \| \tag{6.97}$$

where P is specified in (6.75).

It is an easy matter to verify that in the synthesis of a control system for $\mathcal{P}_x$-process with parameters from (4.2) all the conditions and relationships stated in Theorem 6.8-4 will be fulfilled except (6.67) which essentially is an assumption constraining the class of controlled $\mathcal{P}_x$-processes.

Thus it is proved that given the condition (6.67) is met, for $\mathcal{A}_\mu$ algorithms with inertial COFB the following relationship holds true

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\mu} \left\{ \mathcal{F}_c,\ I_2,\ \frac{c_m}{\delta} \left(\frac{1+\delta+c}{c} \right)^2 \exp \left[\frac{P}{\alpha} \ln \frac{2}{h} \right] \right\} \tag{6.98}$$

Comparing this formula with (6.82) we note that they differ only in the value of the constant N. Consequently, all the remarks and components made in the previous section and following (6.82) apply in this case as well. Accordingly, a general representation on the controlled process behavior in an S_μ system with inertial coordinate-operator feedback may be gleaned from examining Fig. 6.15.

To conclude this topic we note that the qualitative properties of the S_μ systems (6.55)-(6.56) and (6.56), (6.84) are fairly close. The realization of the law (6.84) can be achieved easier than that of (6.55). These control algorithms experience no sliding modes in transient processes, but their CFB and COFB loop gains always exceed those for the integral COFB laws (6.55).

6.9 Integral Coordinate-Operator Feedback with Variable Integration Rate

The binary control algorithms of this work are essentially nonlinear and at the time being the more natural technological basis for their realization is suggested by discrete, specifically microprocessor, hardware. In discretization of binary control system equations used for direct digital control, the preference, of course, should be given to those control algorithms which allow for large steps of integration. Preserving the specified quality of control, this approach is at the same time sparing in the available computing resources. In this section we shall look at one of the possible modific-

ations of control algorithm with integral COFB described in Section 6.7.

Like all other algorithms evaluated in this section, this algorithm also relates to the class of $\mathcal{A}_\mu$ algorithms and is specified by relationships (6.53)-(6.54) and the following differential law of operator variable $\mu(t)$

$$\begin{aligned} \dot{\mu}(t) &= -\alpha \operatorname{sat} \lambda(t) \quad \text{for } |\mu(t)| \leqslant 1 \\ &= -\omega\mu(t) \quad \text{for } |\mu(t)| > 1,\ |\mu(t_0)| \leqslant 1, \quad t \geqslant t_0 \end{aligned} \tag{6.99}$$

Here, sat $(\cdot)$ is given by (6.12); the function $\lambda(t) = \lambda(x(t))$ is defined, as before, on the solutions of the control system as $\lambda(t) = \sigma(x(t))/\delta |x(t)|$, where $\sigma(x)$ is defined by (6.84); α and ω are positive numbers from the set of control algorithm parameters. The dedication and choice of ω have been discussed in Section 6.7 when treating the equations of the integral COFB law (6.55).

Equation (6.99) describes an integral coordinate-operator feedback with variable integration rate. This implies that the aforementioned effect can be achieved by varying the quantization step in a numerical realization of the control algorithm (6.53)-(6.54), (6.99). The limitation $|\mu| \leqslant 1$, required in the problem formulation of the $\mathcal{A}_\mu$ algorithm concerned, is ensured by introducing into the right-hand side of equation (6.99) discontinuities on the manifold $|\mu| = 1$ at which, generally speaking, sliding mode might occur. It will be emphasized again that such sliding modes take place in the controller, rather than in the primary loop closing through the plant. As can be seen from (6.99) and the expression for $\lambda(x)$, the equality $\dot{\mu} = -\alpha \operatorname{sgn} \sigma$ always holds on the boundary of G_δ and beyond it.

The S_μ system resulted with the $\mathcal{A}_\mu$ algorithm of COFB law in $\mathcal{P}_x$-process control is specified by equations (6.56) and (6.99). Its solutions are pairs of functions $(x(t), \mu(t))$ and as usual we shall stick to considering V solutions only. The concept of V solution, its existence and continuity on the entire semiaxis $t \geqslant t_0$ are taken up with reference to the findings of Ref. 93.

A qualitative investigation of the S_μ system (6.56), (6.99) will be conducted in the scheme employed in the two foregoing sections. Because in this case the convergence of controlled processes to zero for $x(t) \in G_\delta$ is also decided by Theorem 6.7-1 and Corollary 6.7-1, the evaluation of the S_μ system (6.56), (6.99) reduces essentially to defining the conditions for G_0 invariance and to proving that the $\mathcal{P}_x$-process is steady $\mathcal{A}_\mu$ controlled. To begin with, we shall elucidate the first subject since it makes up the subject matter of Problem B.

Problem B. For the S_μ system (6.56), (6.99) the relationships ensuring $x(t) \in G_\delta$ for $t \geqslant t_0$ if $x(t_0) \in G_0$ are given by the following theorem.

Theorem 6.9-1. Let $c > 0$ and for the S_μ system (6.56), (6.99)

$$k^0 > \bar{k}^0 \tag{6.100}$$

$$\alpha \geqslant (k^0 + \bar{\bar{k}}^0)\, 8c/\delta \tag{6.101}$$

the constants $\bar{k}^0$ and $\bar{\bar{k}}^0$ being defined by (6.86) and (6.72), respectively. Then the domain G_δ is G_0 invariant for such S_μ system.

Comparison of (6.100) and (6.101) with (6.60) and (6.61) indicates that a change from the integral COFB law of constant integration rate to the integral COFB law with variable integration rate does not inflict a change for the CFB loop gain, while the COFB loop gain increases fourfold. The effect for which the COFB law (6.99) has been suggested does take place.

Proof. Consider for $t \geqslant t_0$ some V solution $(x(t), \mu(t))$ of the S_μ system (6.56), (6.99) such that $x(t_0) \neq O_x$. It can be verified, as before, that for this solution the controlled process $x(t) \neq O_x$ for all $t > t_0$. We wish to demonstrate that if $x(t_0) \in G_0$ then, given (6.100) and (6.101) are met, $x(t) \in G_\delta$ for $t > t_0$.

Assume that at some $t = t'$ the considered process reaches the boundary ∂G_δ of G_δ. Single out an interval $(t_1, t']$ where $\sigma(x(t))$ experiences no sign reversal for the V solution in question and demonstrate that $\mu(t') = -\operatorname{sgn}\sigma(x(t'))$ provided (6.101) holds true. We prove this statement indirectly. Assume that the above is not the case, then from $\sigma(t) > 0$ for all $t \in (t_1, t']$, this assumption yields $\mu(t') > -1$. From (6.99) we have for $\mu(t)$ at $t \in [t_1, t']$

$$\mu(t) = \mu(t_1) - \alpha \int_{t_1}^{t} \lambda(\tau)\, d\tau \tag{6.102}$$

Let q be some number, say $q = 1/2$. In view of the continuity of $\lambda(t)$ on the interval $[t_1, t']$ there is a time instant t_q such that $\lambda(t) \geqslant 1/2$ for t in $[t_q, t']$ and $|T| = t' - t_q > 0$. With these designations we have from (6.102) the obvious equality

$$\mu(t') \leqslant \mu(t_1) - \alpha\, |T|/2 \tag{6.103}$$

Observe now that by assumption $\mu(t') > -1$ and also $\mu(t_1) \leqslant 1$. Then from (6.103) we get an estimate for $|T|$, viz.,

$$|T| < 4/\alpha \tag{6.104}$$

Similar to proving our earlier theorems, here we convince ourselves that on the V solution concerned $(x(t), \mu(t))$ at any t in $[t_1, t']$ for the function $\lambda(t) = \sigma(x(t))/\delta\, |x(t)|$ the integral equation (6.88) is valid. According to the above assumption, $\lambda(t_q) = 1/2$, therefore from (6.88), after estimating the absolute value of the integrand at $t = t'$, we get

$$|\lambda(t')| < 1/2 + (c/\delta)\,(k^0 + \bar{\bar{k}}^0)\,|T|$$

We relax the estimate provided by this inequality by invoking (6.104) and (6.101), viz.,

$$| \lambda (t') | < 1/2 + (4c/\delta\lambda) (k^0 + \bar{\bar{k}}^0) < 1$$

The last inequality in this series is seen to contradict the assumption $| \lambda (t') | = 1$ because $x (t') \in \partial G_\delta$ and, hence, we arrive at the desired equality $\mu (t') = -\text{sgn } \sigma (x (t))$.

To conclude the proof we should repeat the argument from the final section of the proof of Theorem 6.7-2 whence we get that for the V solution concerned $x (t) \in G_\delta$ for all $t \geqslant t_0$. Q.E.D.

By way of comment to Theorem 6.9-1 we note that when its conditions, and those of Theorem 6.7-1 and Corollary 6.7-1 are met, we have for the controlled processes $x (t)$ of the S_μ system (6.56), (6.99) the bound (6.58) at least from the time t_1 when the equality $\sigma (x (t_1)) = 0$ holds for such controlled processes for the first time. The bound (6.58) can be refined for such controlled processes, the refinement is displayed in the following remark proved similar to Remark 6.8-1.

Remark 6.9-1. Let $q = \bar{\bar{k}}^0/k^0$ be taken between 1/2 and 1, and the parameters of the control algorithm (6.53)-(6.54), (6.99) be selected consistent with (6.100) and (6.101). Then for the controlled processes $x (t)$ of the S_μ system (6.56), (6.99) for which $x (t_1) \in G_0$ at all $t \geqslant t_1$ we have $x (t) \in G\delta_q \subset G_\delta$ and

$$\| x (t) \| \leqslant \frac{1+c+\delta_q}{c} \| x (t_1) \| \exp \left[-\frac{1-\delta_q}{c} (t - t_1) \right] \qquad (6.105)$$

where $\delta_q = (1 + q)\, \delta/2$.

Now we would like to touch upon the question of the conditions that would ensure convergence of controlled processes to the domain G_δ. Because answering this question for the S_μ systems (6.56), (6.99) does not entail somewhat significant distinctions from the procedure for the S_μ systems (6.56), (6.84) we shall confine ourselves here to brief remarks. Simultaneously we shall evaluate the asymptotic behavior for those controlled processes which beginning from a certain time find themselves outside G_δ. According to our convention, these topics will be handled under separate problems.

Problems C and D. Consider for $t \geqslant t_0$ a V solution of the S_μ system (6.56), (6.99) such that $x (t_0) \notin G_\delta$. If on the time interval $[t_0, t_1]$ of length $t_1 - t_0 = 2/\alpha$ the function $\sigma (x (t))$ is other than zero on this solution, then for $t \geqslant t_1$ we have $\mu (t) = -\text{sgn } \sigma (x (t))$.

We can easily convince ourselves that this is the case by examining equations (6.99). Therefore for $t \geqslant t_1$ the controlled process $x (t)$ corresponding to this V solution will be determined with the VS system equation (6.65). Since relationship (6.100) is equivalent to (6.60) the findings established for the S_μ system (6.56), (6.84) and

contained in Theorems 6.7-3 and 6.7-4 apply in this case as well. Consequently, inequality (6.60) for $\xi = 0$ and inequality (6.67) ensure G_δ being a domain of attraction for the S_μ system (6.56), (6.99). It is quite obvious also that to those controlled processes $x(t)$ which for $t \geqslant t_1$ manage to reach G_δ, in a finite time, at the moment of reaching t' there corresponds a value of the second component of the respective V solution $(x(t), \mu(t))$, which is defined as $\mu(t') = -\operatorname{sgn} \sigma(x(t'))$. The argument used in proving Theorem 6.9-1 enables us to prove that $x(t) \in G_\delta$ for all subsequent $t \geqslant t'$. If in addition inequality (6.71) is satisfied, then such processes in a finite time appear in G_0 after which under the conditions of Theorem 6.9-1 the bound (6.70) or (6.105) holds true. For convenience of further reference we summarize these considerations in the form of a theorem.

Theorem 6.9-2. Let relationships (6.59), (6.67), (6.71) and (6.101) be valid for the S_μ system (6.56), (6.99) at a constant $c > 0$. Then all the controlled processes in such S_μ system possess the property of $\lim_{t\to\infty} \| x(t) \| = 0$ and divide themselves into two families, $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. Beginning from a certain time, for x^{I} processes, the bound (6.105) is valid.

This theorem defines the conditions of convergence of all controlled processes in the S_μ system (6.56), (6.99) to the origin of the phase space; earlier this property was referred to as the system being $\mathscr{A}_\mu$ controlled. To complete the qualitative study of such S_μ system we need only to establish the fact of stable $\mathscr{A}_\mu$ control and evaluate the dependence of the dynamic properties of controlled processes on the $\mathscr{P}_x$ parameters. Analysis of these topics constitutes the subject of Problems E and F.

Problems E and F. The verification of the property of stable $\mathscr{A}_\mu$ controllability for the S_μ system concerned has a certain salient feature to it. The respective result will be formulated as a lemma analogous to Lemmas 6.7-2 and 6.8-1. This lemma indicates the value for the constant N in inequality (4.4a).

Lemma 6.9-1. Given all the conditions of Theorem 6.9-2 are satisfied, for any controlled process $x(t)$ of the S_μ system (6.56), (6.99) we have for all $t \geqslant t_0$

$$\| x(t) \| \leqslant \frac{c_{\mathrm{m}}}{\delta} \left(\frac{1+\delta+c}{c} \right)^2 \exp(4P/\alpha) \| x(t_0) \| \tag{6.106}$$

where the constant P is given by (6.74), and $c_{\mathrm{m}} = \max(c, 1)$.

Proof. We prove the lemma in two steps. First, we strengthen the result established in Theorem 6.9-1. Essentially this strengthening concerns the fact that under the conditions considered G_δ is in fact $G_{\delta/2}$ invariant for the S_μ system (6.56), (6.99). Then we resort to the argument of the proof of Lemma 6.7-2.

Step 1. Let $(x(t), \mu(t))$ be a V solution of the S_μ system under investigation and such that $x(t_0) \in G_{\delta/2}$ and $|\mu(t_0)| \leqslant 1$. The function $\lambda(t) = \sigma(x(t))/\delta \,|\, x(t) \,|$ varies for $t \geqslant t_0$ along such a solution as

$$\lambda(t) = \lambda(t_0) + \frac{c}{\delta} \int_{t_0}^{t} [(a_1 - a_2\zeta + \zeta^2) \operatorname{sgn} x + k^0\mu] \, d\tau \qquad (6.107)$$

where $\zeta = (1 + \delta\xi)/c$ and $\xi = \lambda(t) \operatorname{sgn} x(t)$.

Assume that at some $t' > t_0$ we have $|\lambda(t')| = 1$ — this corresponds to the controlled process appearing on the boundary of G_δ. Then owing to the continuity of $\lambda(t)$ there exists a time instant $t_1 > t_0$ such that $|\lambda(t)| > 1/2$ for all $t \in (t_1, t']$. From equation (6.107) it can be seen (see also Remark 6.9-1) that $|\lambda(t)|$ may increase, generally speaking, subject to the inequality

$$-\mu(t) \operatorname{sgn} \sigma(x(t)) \leqslant q$$

where $q = \bar{\bar{k}}^0/k^0$ and $1/2 < q < 1$ (notice that $\operatorname{sgn} \lambda = \operatorname{sgn} \sigma$ for $t \in (t_1, t']$). Making use of (6.102), as in the proof of Theorem 6.9-1, we obtain for $|T| = t' - t_1$

$$|T| < 2(1 + q)/\alpha$$

Considering now expression (6.107) for $t \geqslant t_1$ and using the above derived inequality for $|T|$ we get, under the assumptions made above, the following sequence of inequalities:

$$|\lambda(t')| \leqslant \frac{1}{2} + \frac{c}{\delta}(k^0 + \bar{\bar{k}}^0)\,|T| < \frac{3+q}{4} < 1$$

the last of which contradicts the assumption of $|\lambda(t')| = 1$. Therefore, for all $t \geqslant t_0$ $|\lambda(t)| < 1$ and, hence, the domain G_δ is $G_{\delta/2}$ invariant.

Step 2. Consider a V solution of the S_μ system (6.56), (6.99) for $t \geqslant t_0$. Assume $t_1 = t_0 + 4/\alpha$. If at some t' in $[t_0, t_1]$ the inequality $|\lambda(t')| \leqslant 1/2$ is satisfied, then, as has been indicated above, $|\lambda(t)| < 1$ for all $t \geqslant t'$ and, hence, the estimate (6.58) will be true. If for the V solution considered $|\lambda(t)| > 1/2$, then at $t = t_1$ we face two situations: either $x(t_1) \in G_\delta$, or $x(t_1) \notin G_\delta$, for both situations $\mu(t_1) = -\operatorname{sgn} \sigma(x(t_1))$.

For the first situation, from Theorem 6.9-2 the controlled process $x(t) \in G_\delta$ for all $t > t_1$ and inequality (6.58) is met. If the other situation takes place, then it is examined like case 2 of the proof for Lemma 6.7-2. In all situations, the use of the first inequality from the sequence (6.77) at $t = t_1$ produces the desired estimate (6.106). Lemma 6.9-1 is proved.

We have proved that under the conditions of Theorem 6.9-2 the parametrically ill-defined $\mathcal{P}_x$-process is steady $\mathcal{A}_\mu$ controlled with the algorithm (6.53)-(6.54), (6.99). Because all the relationships of

the theorem, except (6.67), can always be satisfied by a suitable choice of control algorithm parameters, it may be deemed established that given (6.67) is met, the following relationship holds for the said $\mathcal{A}_\mu$ algorithm

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\mu} \left\{ \mathcal{F}_c, I_2; \frac{c_m}{\delta} \left(\frac{1+\delta+c}{c} \right)^2 \exp \frac{4P}{\alpha} \right\} \tag{6.108}$$

This relationship is equivalent to (6.82) and (6.89), differing from these by the content of the constant N. Therefore, all the remarks and conclusions outlined in detail in Problem E of Section 6.7 relate to the properties of controlled processes in the S_μ system (6.56), (6.99). They also concern the dependence of the dynamic properties of controlled processes on the $\mathcal{P}_x$ parameters or system's structural parameters and the possibility to achieve an asymptotically independent behavior with respect to parametric disturbances. A qualitative insight into the nature of transient processes in the S_μ system (6.56), (6.99) can be obtained from examining Fig. 6.15.

6.10 Continuous Inertial Coordinate-Operator Feedback

Under certain conditions the $\mathcal{A}_\mu$ algorithm with COFB (6.99) treated in the previous section enables numerical integration of closed-loop system equations to be done with a greater step than for the equations of S_μ system with the $\mathcal{A}_\mu$ algorithm (6.53)-(6.55). The respective integration of S_μ systems with COFB (6.99) and with COFB (6.55) may be effected accurate only to the first order of accuracy. This is associated with the fact that in such S_μ systems there is a likelihood of sliding modes occurring in the auxiliary control loop on the sets $|\mu| = 1$. The $\mathcal{A}_\mu$ algorithm with inertial COFB devised in Section 6.8 prevents such sliding modes from occurring. In this connection it appears reasonable to consider a control algorithm which would unite the advantages of the control algorithms evaluated in Sections 6.8-6.9. The ensuing section will be devoted exactly to such an algorithm.

This control algorithm is given by relationships (6.53)-(6.54) and the following differential law for the operator variable $\mu(t)$:

$$\dot{\mu}(t) = -\alpha [\mu(t) + \operatorname{sat} \lambda(t)], \quad t \geqslant t_0 \tag{6.109}$$

In what follows we shall refer to this COFB law as a continuous inertial law.

The behavior of the S_μ system with the said control algorithm is defined by equations (6.56) and (6.109) with continuous right-hand sides. The properties of solutions are completely elucidated within

the classical theory of differential equations. These systems of differential equations can be numerically integrated using traditional difference techniques of high order.

Within the framework of the analysis scheme outlined above we wish now to investigate the S_μ system (6.56), (6.109), treating in depth only those system properties which differ substantially from the dynamics of other S_μ systems and limiting ourselves to brief comments in other cases.

A solution of the S_μ system (6.56), (6.109), as the solutions of all closed-loop systems with dynamic COFB laws, is constituted by a pair of functions $(x(t), \mu(t))$. Of all the system solutions we shall consider only V solutions which correspond to the initial conditions (x_0, μ_0) of the set $V = \{(x, \mu): x \in \mathbb{R}^2, |\mu| \leqslant 1\}$. We shall focus our analysis in particular on the V' solutions corresponding to the initial conditions from the set $V' = \{(x, \mu): x \in \mathbb{R}^2, |\mu| \leqslant 1\}$. The possibility to limit our discussion without loss of generality to evaluating only the properties of such solutions has been taken up in depth in Section 6.8.

First of all we wish to clear out the conditions under which the controlled processes $x(t)$ in the S_μ system (6.56), (6.109) belong to the domain G_δ.

Problem B. Like with all S_μ systems with dynamic COFB the property $x(t) \in G_\delta$ may be warranted for the controlled processes taking off from G_0. Definition 6.7-1 refers to this property as G_0 invariance. The sufficient conditions for the domain G_δ to be G_0 invariant are defined by the following theorem.

Theorem 6.10-1. Let h be a number between 1/2 and 1, c be a positive constant, and for the S_μ system (6.56), (6.109)

$$k^0 > \bar{k}^0/(1-h) \tag{6.110}$$

$$\alpha \geqslant \frac{2c}{\delta}(k^0 + \bar{\bar{k}}^0) \ln \frac{4}{2h-1} \tag{6.111}$$

where the constants $\bar{k}^0$ and $\bar{\bar{k}}^0$ are calculated in accord with (6.86) and (6.72). Then the domain G_δ is G_0 invariant for such S_μ system.

Relationships (6.110) and (6.111) determine the CFB and COFB loop gains. Comparison of (6.110) and (6.111) with (6.100) and (6.101) indicates that the incorporation of a continuous inertial COFB means that these gains have to be increased. It will be recalled that choice of parameters k^0 and α for the control algorithm is a matter of trade-off achieved by a suitable specification of constant h.

Proof. We shall only lay out the scheme of the proof, respective details may be reconstructed by invoking the argument from the proofs for Theorem 6.8-1 and 6.9-1.

We consider for $t \geqslant t_0$ a V' solution $(x(t), \mu(t))$ such that $\lambda(t_0) = 0$. The last equality implies that $\sigma(x(t_0)) = 0$. Assuming that rela-

tionships (6.110)-(6.111) hold true and on the said V' solution at some $t = t_2 > t_0$ the controlled process reaches the boundary ∂G_δ of G_δ, i.e., $|\lambda(t_2)| = 1$, we demonstrate, as in proving Theorem 6.9-1, that $-\mu(t_2)\operatorname{sgn}\lambda(t_2) \geqslant 1 - h$. Indeed, from the continuity of $\lambda(t)$ we infer that there exists a time interval $[t_1, t_2]$, $t_1 < t_2$, on which $|\lambda(t)| \geqslant 1/2$. On this interval the variation of $\mu(t)$, according to (6.109), is governed by the equation

$$\mu(t) = \mu(t_1)\, e^{-\alpha(t-t_1)} - \alpha \int_{t_1}^{t} e^{-\alpha(t-\tau)} \lambda(\tau)\, d\tau$$

With account of the inequality $|\lambda(t)| > 1/2$ satisfied for all t in $(t_1, t_2]$ and assumption of the inequality $-\mu(t_2)\operatorname{sgn}\lambda(t_2) < 1 - h$ contradicting that required, we can readily obtain the length bound for the interval $[t_1, t_2]$, viz.,

$$t_2 - t_1 < \frac{1}{2} \ln \frac{4}{2h-1} \tag{6.112}$$

Then we use the integral equation (6.95) to estimate the variation of $\lambda(t)$ on the interval $[t_1, t_2]$ and, observing (6.112), arrive for $t = t_2$ at the following sequence of inequalities (similar to proving Theorem 6.9-1):

$$\lambda(t_2) \leqslant \frac{1}{2} + \frac{c}{\delta}(k^0 + \bar{\bar{k}}^0)(t_2 - t_1) < \frac{1}{2} + \frac{c(k^0 + \bar{\bar{k}}^0)}{\alpha\delta} \ln \frac{4}{2h-1}$$

From the last inequality and (6.111) we get $|\lambda(t_2)| < 1$, and that contradicts the assumption of $x(t_2) \in \partial G_\delta$. Consequently, we arrive at the desired inequality $-\mu(t_2)\operatorname{sgn}\lambda(t_2) \geqslant 1 - h$. Further application of the argument from the proof of Theorem 6.8-1 reveals that if (6.110) is true then $x(t) \in G_\delta$ for all $t \geqslant t_0$. Q.E.D.

To summarize, if the parameters of the $\mathscr{A}_\mu$ algorithm with COFB (6.109) are selected in accord with the conditions of Theorems 6.7-1 and 6.10-1 and Corollary 6.7-1, then the controlled processes taking off in the domain G_0 converge to the origin of the phase space and obey the exponential bound (6.58). With a sufficiently small value of δ we may speak of a weak dependence of the dynamic properties of such controlled processes on $\mathscr{P}_x$ parameters arbitrarily varying within the ranges (4.2). It is worth emphasizing that in the S_μ system considered this property can be achieved with smooth control laws and finite gains.

Suppose that a choice of system's structural parameters is made in one way or another, then the bound (6.58) for the S_μ system (6.56), (6.109) can be refined. This refinement is contained in the following remark, similar to Remarks 6.8-1 and 6.9-1 made for other S_μ systems.

Remark 6.10-1. Assume relationships (6.110), (6.111) hold at some h taken between 1/2 and 1, and $(1 - h)\, k^0 > \overline{\overline{k}}^0$. Then if $h = 1 - \overline{\overline{k}}^0/k^0$ and for $t \geqslant t_1$ $x(t) \in G_{\delta_h} \subset G_\delta$, we have

$$\|x(t)\| \leqslant \frac{1+c+\delta_h}{c} \|x(t_1)\| \exp\left[-\frac{1-\delta_h}{c}(t-t_1)\right] \tag{6.113}$$

where $\delta_h = \delta\varphi(h)$, and $\varphi(h) = \ln[4/(2h-1)]/\ln[4/(2h-1)] < 1$.

We omit the proof of this remark as this can be readily obtained from the reasoning outlined in proving Theorem 6.10-1.

What remains to complete our qualitative study of the S_μ system (6.56), (6.109) is to establish convergence to the phase space origin for those controlled processes x which do not belong to G_δ, and also the fact of stable $\mathcal{A}_\mu$ controllability of $\mathcal{P}_x$-processes with the control algorithm (6.53)-(6.54), (6.109).

Problems D and E. For the S_μ system (6.56), (6.109), the conditions for G_δ attraction, asymptotic behavior of controlled processes $x(t) \notin G_\delta$, and the fact of stable $\mathcal{A}_\mu$ controllability of $\mathcal{P}_x$-processes can be evaluated in much the same way as in Section 6.8 for the S_μ system (6.56), (6.84). Also, direct analogues of Theorems 6.8-2 and 6.8-3 and Lemma 6.8-1 can be formulated. Their proofs will differ from the origins only insignificantly, therefore we omit them here, limiting ourselves only to phrasing the ultimate statement on the properties of the controlled processes $x(t)$ in the S_μ system studied.

Theorem 6.10-2. Let relationships (6.59), (6.67), (6.111) and (6.96) hold for the system S_μ at some positive constants c and h, $1/2 < h < 1$. Then all the controlled processes exhibit the property of $\lim_{t\to\infty} \|x(t)\| = 0$ and divide themselves in two families, $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. Each x^{I} process may be bounded for all $t \geqslant t_0$ as

$$\|x(t)\| \leqslant \frac{c_m}{\delta}\left(\frac{1+\delta+c}{c}\right)^2 \exp\left(\frac{P}{\alpha}\ln\frac{4}{2h-1}\right)\|x(t_0)\| \tag{6.114}$$

where the expression for P in terms of system parameters is given by (6.74). For the x^{II} processes, the bound (6.113) is valid beginning from some time instant $t_1 > t_0$.

This theorem defines the conditions of convergence for all considered controlled processes in the S_μ system (6.56), (6.109). In line with the adopted concepts this implies that the $\mathcal{P}_x$-process is $\mathcal{A}_\mu$ controlled by the control algorithm (6.53)-(6.54), (6.109). The theorem asserts also that this $\mathcal{A}_\mu$ controllability is stable and indicates the constant N appearing in (4.4*a*).

It will be noted now that all the relationships indicated in Theorem 6.10-2, except the condition (6.67), can be satisfied in control system design. Therefore, the following property of $\mathcal{A}_\mu$ algorithms (6.53)-(6.54), (6.109) may be thought of as being proved.

Given the condition (6.67) is met, the following relationship is valid for $\mathscr{A}_\mu$ algorithms with continuous inertial COFB:

$$\mathscr{P}_x \in \mathscr{C}\mathscr{A}_\mu \{\mathscr{F}_c,\ I_2;\ \frac{c_m}{\delta}\left(\frac{1+\delta+c}{c}\right)^2 \exp\left[\frac{P}{\alpha}\ln\frac{4}{2h-1}\right]\} \qquad (6.115)$$

Since this expression differs from (6.82) in the value of N only, the general properties of S_μ systems with COFB (6.109) are similar to those of S_μ systems with COFB (6.55) and, therefore, all remarks made above in connection with the analysis of (6.82) apply here as well. A qualitative insight as to the character of transient processes in the S_μ systems (6.56), (6.109) at some fixed parameters of the $\mathscr{P}_x$-process can, in this case also, be gleaned from inspecting Fig. 6.15.

In summary, we have demonstrated that with allowance for some increase for CFB and COFB loop gains, the designer may change from discontinuous COFB laws to continuous laws retaining all the positive properties, typical of S_μ systems with discontinuous COFB laws. This transition is attracting by the appearing possibility to employ ordinary difference schemes of desired accuracy for the numerical integration of such continuous dynamic systems. This advantage may prove significant in exploiting such control algorithms in direct digital control systems.

6.11 S_μ (2) Systems Under Integral Coordinate-Operator Feedback Law

For the S_μ systems treated in the foregoing sections, the property of stable $\mathscr{A}_\mu$ controllability has been proved on the assumption the condition (6.67) is met. Of course, this assumption limits the class of $\mathscr{P}_x$-processes for which stable $\mathscr{A}_\mu$ control is available. In what follows we will attempt to lift this limitation. The approach suggested for this purpose consists in changing from $\mathscr{A}_\mu$ (1) algorithms to $\mathscr{A}_\mu$ (2) algorithms, the number "2" in parentheses meaning that the CFB law employs direct action not only in the $x(t)$ coordinate, as is the case with $\mathscr{A}_\mu$ (1) algorithms, but also in the $\dot{x}(t)$ coordinate. This leads to $\mathscr{A}_\mu$ (2) algorithms and, respectively, to closed-loop S_μ (2) systems.

Of course, transfer to $\mathscr{A}_\mu$ (2) algorithms complicates the controller but it is attractive in a number of positive properties it brings along. One of them we have already mentioned, the other is associated with substantial simplification of system design as the conditions of invariance for G_δ domains take on an especially simple form in such cases.

We shall evaluate the properties of systems with $\mathscr{A}_\mu$ (2) control algorithms for the integral COFB law only. Systems with other COFB laws may be handled in a similar manner. To avoid confusion we

note from the very outset of our consideration that a complete solution of the control problem is inachievable in S_μ (2) systems. That is to say, a weak dependence of system dynamics on $\mathcal{P}_x$ parameters can be ensured only for a portion of controlled processes in the class of continuous control actions, while asymptotically independent response to parametric disturbances is absent altogether. However, as before, a suitable choice of control algorithm parameters may drive the system arbitrarily close to the property (6.51), though at the expense of possible increase of COFB loop gain.

Now we demonstrate the relationships defining the $\mathcal{A}_\mu$ (2) algorithm with integral COFB under investigation. The control in the primary (CFB) control loop is generated as

$$u(t) = k_1(t) \mid x(t) \mid + k_2(t) \mid \dot{x}(t) \mid \tag{6.116}$$

where the functions $k_1(t)$ and $k_2(t)$ representing the gains depend upon the operator variable $\mu(t)$ and are defined as follows:

$$k_i(t) = k_i^0 \mu(t), \quad i = 1, 2 \tag{6.117}$$

Here k_i^0 are positive constants entering the set of $\mathcal{A}_\mu$ (2) algorithm circuit parameters.

Since only integral coordinate-operator feedback will be treated, the variation law for the operator variable $\mu(t)$ must have the form (6.55). We repeat this law for convenience:

$$\begin{aligned} \dot{\mu}(t) &= -\alpha \operatorname{sgn} \sigma(x(t)) \quad \text{for } |\mu(t)| \leqslant 1 \\ &= -\omega\mu(t) \quad \text{for } |\mu(t)| > 1, \ |\mu(t_0)| \leqslant 1, \ t \geqslant t_0 \end{aligned}$$

Substituting (6.116) and (6.117) into the equation for $\mathcal{P}_x$-process (Chapter 4) we get the following differential equation in $x(t)$:

$$\ddot{x}(t) + a_2(t)\dot{x}(t) + a_1(t)x(t) = -k_1^0\mu(t)|x(t)| - k_2^0\mu(t)|\dot{x}(t)|, \quad t \geqslant t_0 \tag{6.118}$$

Equations (6.55) and (6.118) and inequalities (4.2) completely define the S_μ (2) system on hand. As usual, the solutions to such a control system will be given by pairs of functions such as $(x(t), \mu(t))$. As for the notion of solution to S_μ system (6.58), (6.118), its existence, continuity into the semiaxis $t \geqslant t_0$, and behavior, all remarks made in Section 6.7 for S_μ (1) system solutions apply here as well.

Because in our further explication of results we wish to recourse to Definition 6.7-3, we give the equation of VS (2) system tied with the S_μ (2) system (6.55), (6.118), viz.,

$$\begin{aligned} &\ddot{x}(t) + a_2(t)\dot{x}(t) + a_1(t)x(t) \\ &= [k_1^0 \mid x(t) \mid + k_2^0 \mid \dot{x}(t) \mid] \operatorname{sgn} \sigma(x(t)), \quad t \geqslant t_0 \end{aligned} \tag{6.119}$$

where $\sigma(x(t)) = -cx(t)$, $c = (1, c)$, and the functions $a_1(t)$ and $a_2(t)$ satisfy (4.2).

In references to Definition 6.7-3 we shall also assume that in its formulation the term VS system is replaced with VS (2) system. We note also that in the qualitative study of the S_μ (2) system (6.55), (6.118) we shall confine ourselves to evaluating its V solutions only.

Like in similar situations above, we convince ourselves that it is generally impossible to keep the exact equality $\sigma(x(t)) = 0$ on S_μ (2) system solution beginning from a certain time instant. Therefore, like in Section 6.7, the choice of structural parameters k_1^0, k_2^0, α, δ, and c of the $\mathcal{A}_\mu$ (2) control algorithm is aimed at an approximate fulfillment of this equality. In this section this approach will be understood in the sense of $x(t) \in \overline{G}_\delta$ (at least for some controlled processes $x(t)$), where the domain $\overline{G}_\delta \subset \mathbb{R}_x^2$ is given as

$$\overline{G}_\delta = \{x: |\sigma(x)| \leqslant \delta\beta |x|\} \tag{6.120}$$

where $\beta = [\beta_1, \beta_2]$ is a row vector of constant components $0 < \beta_i \leqslant 1$, $i = 1, 2$; $|x| = [x, \dot{x}]^T$, and δ is a positive constant. The boundary of the domain $\overline{G}_\delta$ will be denoted as $\partial\overline{G}_\delta$ and defined as

$$\partial\overline{G}_\delta = \{x: [\sigma^+(x) = \sigma(x) - \delta\beta|x| = 0] \cup [\sigma^-(x) = \sigma(x) + \delta\beta|x| = 0]\} \tag{6.121}$$

The geometrical pattern of the domain $\overline{G}_\delta$ coincides with that of the domain G_δ indicated in Fig. 6.7.

For convenience, we convert equation (6.118) in the vector-matrix form

$$\dot{x}(t) = A(t)x(t) - \mu(t)bk^0|x(t)|, \quad t \geqslant t_0 \tag{6.122}$$

by denoting $A(t) = [a^1, a^2]$ the 2×2 matrix of column vectors $a^1(t) = [0, -a_1(t)]^T$ and $a^2(t) = [1, -a_2(t)]^T$; $b = [0, 1]^T$, and $k^0 = [k_1^0, k_2^0]$. In this notation the control function $u(t)$ becomes

$$u(t) = \mu(t)k^0|x(t)|$$

and equation (6.119) of the VS (2) system rewrites as

$$\dot{x}(t) = A(t)x(t) + \mu(t)bk^0|x(t)|\operatorname{sgn}\sigma x(t), \quad t \geqslant t_0$$

Finally, we tie up with the $\mathcal{P}_x$-process on hand some constants which appear in statements to be phrased below. For a certain $c > 0$ we introduce six constants, viz., $\tilde{k}_i^0$, $\bar{k}_i^0$, and $\bar{\bar{k}}_i^0$, $i = 1, 2$, as follows:

$$\tilde{k}_i^0 = \sup_{t \geqslant t_0} |ca^i(t)|/c, \quad i = 1, 2 \tag{6.123a}$$

$$\bar{k}_i^0 = \max_{\tilde{c} \in \Sigma_1} \sup_{t \geqslant t_0} |\tilde{c}a^i(t)|/|\tilde{c}b|, \quad i = 1, 2 \tag{6.123b}$$

where $\Sigma_1 = \{\tilde{c}:\tilde{c} = c + \xi,\ \xi = [\xi_1, \xi_2],\ \xi_i = \pm \delta\beta_i.\ i = 1, 2\}$;

$$\bar{\bar{k}}_i^0 = \max_{\tilde{c} \in \Sigma_2} \sup_{t \geqslant t_0} |\tilde{c}a^i(t)|/|\tilde{c}b|,\ i = 1, 2 \qquad (6.123c)$$

where $\Sigma_2 = \{\tilde{c}:\tilde{c} = c + \lambda\xi,\ |\lambda| \leqslant 1,\ \xi = [\xi_1, \xi_2],\ \xi_i = \pm\delta\beta_i,\ i = 1, 2\}$

From examining (6.123) it is not hard to see that whenever these expressions take sense, the knowledge of the variation ranges for the parameters a_1 and a_2 of the $\mathcal{P}_x$-process is sufficient to compute them. Comparison of (6.123) with (6.72) and (6.86) suggests also that the constants $\bar{k}_i^0$ and $\bar{\bar{k}}_i^0$, $i = 1, 2$, are simpler to compute than $\bar{k}^0$ and $\bar{\bar{k}}^0$ from (6.86) and (6.72).

In the qualitative study of the S_μ (2) system (6.55), (6.122) we shall adhere to the scheme of analysis developed in Section 6.7 and consisting of solving a set of Problems A through F. This investigation will be displayed in detail, for it differs markedly from the cases evaluated earlier. We begin our S_μ (2) system analysis by solving Problem A.

Problem A. It deals with the asymptotic properties of the controlled processes $x(t) \in \bar{G}_\delta$. Similarly to Section 6.7, the equation (differential constraint)

$$\dot{x}(t) = -\frac{1}{c}x(t) - \frac{1}{c}\sigma(t)$$

will be pivotal for the analysis.

Used together with the inequality $|\sigma(t)| \leqslant \delta\beta |x(t)|$ valid whenever $x(t) \in \bar{G}_\delta$, this equation yields the bounds for the norms of controlled processes under the considered conditions and pinpoints the conditions for these processes to converge to the phase space origin.

The formulation and proof of key statements will be preceded by deriving some bounds. We note that for $x \in \bar{G}_\delta$, $|\dot{x}| \leqslant |x|/c + |\sigma|/c \leqslant |x|/c + \|x\| \delta\beta_m/c$,

where $\beta_m = \max(\beta_1, \beta_2)$.

Using this inequality we find for $x \in \bar{G}_\delta$ the bound for the norm $\|x\| = |\dot{x}| + |x|$, viz.,

$$\|x\| \leqslant |x|(c + 1)/(c - \delta\beta_m) \qquad (6.124)$$

which is valid subject to a constant δ being such that $c - \delta\beta_m > 0$. Accordingly, for $x \in \bar{G}_\delta$ we have another bound $|\sigma(x)| \leqslant \delta\beta_m |x| (c + 1)/(c - \delta\beta_m)$.

Now that we have finished with the new notation and the preparatory inequalities, we are in a position to prove the following theorem on the norms of controlled processes.

Theorem 6.11-1. Assume that at some positive constants c and δ

$$\delta\beta_m < c \tag{6.125}$$

and $x(t) \in \bar{G}_\delta$ for some V solution $(x(t), \mu(t))$ of S_μ (2) system for all t in $[t_1, t_2]$, $t_2 > t_1 \geqslant t_0$. Then for this V solution at all t in $[t_1, t_2]$

$$\| x(t) \| \leqslant \frac{c+1+\bar{\delta}}{c} \| x(t_1) \| \exp\left[-\frac{1-\bar{\delta}}{c} (t-t_1) \right] \tag{6.126}$$

where the constant $\bar{\delta}$ is given by

$$\bar{\delta} = \delta\beta_m (c+1)/(c-\delta\beta_m) \tag{6.127}$$

We omit the proof for this theorem as being traditional for the theory of differential equations.

Comparing the statements of Theorems 6.7-1 and 6.11-1 we see that the S_μ (2) system case contains an additional condition (6.125) due to specifying the domain $\bar{G}_\delta$ in the form (6.120). This condition is not hard to meet by a suitable choice of δ or β_i, $i = 1, 2$. It will be noted that inequality (6.125) warrants the boundness of $\bar{k}_i^0$, $\bar{\bar{k}}_i^0$, $i = 1, 2$, and the positive sign of the product $\boldsymbol{c} \cdot \boldsymbol{b}$.

Observing the bound (6.126), inequality (6.125), and expression (6.127) it will be an easy matter to prove the corollary of Theorem 6.11-1 which defines the conditions for controlled processes to converge to the origin of the phase space.

Corollary 6.11-1. Let all the conditions of Theorem 6.11-1 be met, $t_2 = \infty$ and

$$\delta < c/\beta_m (c+2) \tag{6.128}$$

Then for any V solution of the S_μ (2) system (6.55), (6.122) we have $\lim_{t \to \infty} \| x(t) \| = 0$.

Determine now the conditions under which $x(t) \in \bar{G}_\delta$ for some V solutions of the S_μ (2) system (6.55), (6.122). Evaluation of these conditions makes up the subject of Problem B.

Problem B. Its solution is given by the following theorem.

Theorem 6.11-2. Let for the S_μ (2) system (6.55), (6.122) there be valid inequality (6.125) and the relationships

$$k_i^0 > \bar{k}_i^0, \quad i = 1, 2 \tag{6.129}$$

$$\alpha \geqslant \max_{i=1, 2;\ \tilde{c} \in \Sigma_2} 2\tilde{c}\boldsymbol{b} (k_i^0 + \bar{\bar{k}}_i^0)/\delta\beta_i \tag{6.130}$$

where the set Σ_2 is defined in (6.123c). Then the domain $\overline{G}_\delta$ is G_0 invariant for this S_μ (2) system.

Relationships (6.129)-(6.130) determine the choice of gains in the CFB and COFB loops. They are seen to be simpler than the respective relationships for S_μ (1) systems. It is worth noting also that given inequality (6.125) is met, the product $\boldsymbol{c}\cdot\boldsymbol{b}$ is strictly positive.

Proof. We omit the details of the proof as they can be readily reconstructed from the proof of Theorem 6.7-2, confining ourselves to displaying only the key relationships.

For the V solutions at hand we introduce the function $\lambda(t) = \sigma(x(t))/\delta\beta \mid x(t) \mid$ varying for $t \geqslant t_1$ on a V solution as defined by

$$\lambda(t) = \lambda(t_1) + \int_{t_1}^{t} \frac{\dot{\sigma}(\tau) - \lambda(\tau)\,\delta\beta\,|x(\tau)|^{\cdot}}{\delta\beta\,|x(\tau)|}\,d\tau$$

$$= \lambda(t_1) + \int_{t_1}^{t} \frac{-\tilde{\boldsymbol{c}}\,(\boldsymbol{A}S^x - \mu\boldsymbol{b}\boldsymbol{k}^0)\,|x|}{\delta\beta\,|x(t)|}\,d\tau \qquad (6.131)$$

where $\tilde{\boldsymbol{c}} = \boldsymbol{c} + \lambda\delta\beta S^x$, $S^x(t)$ is the 2×2 diagonal matrix of the elements $s_{11} = \operatorname{sgn} x(t)$ and $s_{22}(t) = \operatorname{sgn} \dot{x}(t)$. This equation has been derived with the use of equation (6.122). Changing to scalar form in the integrand, after some algebra we get

$$\lambda(t) = \lambda(t_1) + \int_{t_1}^{t} \frac{[\mu\tilde{\boldsymbol{c}}\boldsymbol{b}k_1^0 - \tilde{\boldsymbol{c}}\boldsymbol{a}^1 \operatorname{sgn} x,\ \mu\tilde{\boldsymbol{c}}\boldsymbol{b}k_2^0 - \tilde{\boldsymbol{c}}\boldsymbol{a}^2 \operatorname{sgn} \dot{x}]\,|x|}{\delta\beta\,|x|}\,d\tau \qquad (6.132)$$

Noting that the maximum of the integrand never exceeds the absolute-value ratios of the coefficients of $|x|$ and $|\dot{x}|$ in numerator and denominator, we get

$$|\lambda(t)| \leqslant |\lambda(t_1)| + \int_{t_1}^{t} \max\left(\frac{|\mu\tilde{\boldsymbol{c}}\boldsymbol{b}k_1^0 - \tilde{\boldsymbol{c}}\boldsymbol{a}^1 \operatorname{sgn} x|}{\delta\beta_1},\ \frac{|\mu\tilde{\boldsymbol{c}}\boldsymbol{b}k_2^0 - \tilde{\boldsymbol{c}}\boldsymbol{a}^2 \operatorname{sgn} \dot{x}|}{\delta\beta_2}\right) d\tau$$

After an obvious relation this bound takes on the form

$$|\lambda(t)| \leqslant |\lambda(t_1)| + \int_{t_1}^{t} \max_{i=1,2} \frac{|\tilde{\boldsymbol{c}}\boldsymbol{b}|}{\delta\beta_i}\left(k_i^0 + \frac{|\tilde{\boldsymbol{c}}\boldsymbol{a}^i|}{|\tilde{\boldsymbol{c}}\boldsymbol{b}|}\right) d\tau$$

$$\leqslant |\lambda(t_1)| + \int_{t_1}^{t} \max_{i=1,2} \frac{|\tilde{\boldsymbol{c}}\boldsymbol{b}|}{\delta\beta_i}(k_i^0 + \overline{\overline{k}}_i^0)\,d\tau \qquad (6.133)$$

where the constants $\overline{\overline{k}}_i^0$, $i = 1, 2$, are defined in (6.123c).

Assuming now that at some time instant $t' > t_1$ the controlled process at hand reaches the boundary of $\overline{G}_\delta$, i.e. $x(t') \in \partial\overline{G}_\delta$, we prove that, as in Theorem 6.7-2, at this time instant for the second

component of the V solution $\mu(t') = -\operatorname{sgn}\sigma(\boldsymbol{x}(t'))$. To this end, we invoke inequalities (6.133), (6.130) and the fact that $\lambda(t_1) = 0$, i.e. the V solution takes off from the domain G_0. Further we take up equation (6.132) to examine it at $t = t'$ with account of the proved equality $\mu(t') = -\operatorname{sgn}\lambda(t')$, $|\lambda(t')| = 1$, inequality (6.129), and the fact that given (6.125) is met $\widetilde{cb} > 0$ for all $t \geqslant t_1$. As a result, we find out that $|\lambda(t)|$ falls off for $t > t'$ which means that $\boldsymbol{x}(t) \in \overline{G_\delta}$, thus completing the proof.

Thus, given the conditions of Theorems 6.11-1 and 6.11-2 are satisfied, any controlled process starting from G_0 will converge to the origin of the phase space. Now we wish to look at conditions under which any controlled process converges to zero, i.e., the $\mathcal{P}_x$-process is $\mathcal{A}_\mu$ controlled. Evaluation of these conditions constitutes the subject of Problems C and D.

Problems C and D. For the S_μ (2) system at hand the conditions of $\overline{G_\delta}$ attraction are given by Theorem 6.7-3 and the following statement, substituting in this case for Theorem 6.7-4.

Theorem 6.11-3. Let for a constant $c > 0$

$$k_i^0 > \bar{k}_i^0, \quad i = 1, 2 \tag{6.134}$$

Then the domain G_0 is $\mathcal{O}$-attracting for the VS system.

The proof of this theorem is straightforward and immediately follows from examining the expression for the derivative $\dot{\sigma} = \dot{\sigma}(\boldsymbol{x}(t))$ obtained with the use of equation (6.122), viz.,

$$\begin{aligned}\dot{\sigma} &= -\boldsymbol{c}\dot{\boldsymbol{x}} = -\boldsymbol{c}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}k^0|\boldsymbol{x}|\operatorname{sgn}\sigma)\\ &= -(ck_1^0\operatorname{sgn}\sigma + \boldsymbol{c}\boldsymbol{a}^1\operatorname{sgn}x)|x| - (ck_2^0\operatorname{sgn}\sigma + \boldsymbol{c}\boldsymbol{a}^2\operatorname{sgn}\dot{x})|\dot{x}|\end{aligned}$$

Taking into account relationships (6.134) and (6.123a), and then Lemma 6.7-1 on the right-hand side of the above expression yields the proof for Theorem 6.11-3.

Comparing Theorem 6.11-3 with similar theorems for S_μ (1) systems we note that S_μ (2) systems no longer need the assumption (6.67) to be met, accordingly, $\mathcal{A}_\mu$ (2) algorithms are applicable for control over a wider class of $\mathcal{P}_x$-processes than $\mathcal{A}_\mu$ (1) algorithms.

Put into a more tractable perspective, Theorem 6.11-3 states that under the aforestated conditions two types of control process behavior are possible in S_μ (2) systems. For one type, $\boldsymbol{x}^{\mathrm{I}}(t)$, of the process the attainment of the domain $\overline{G_\delta}$ is ensured beginning from a certain finite time instant. As for the other type, $\boldsymbol{x}^{\mathrm{II}}(t)$ processes, from some finite time instant they belong to the area $\mathbb{R}_x^2 \setminus \overline{G_\delta}$ (where as usual the slant line means "less than") and $\lim_{t\to\infty} \|\boldsymbol{x}^{\mathrm{II}}(t)\| = 0$.

Using equation (6.132) it is not hard to verify, like in Section 6.7 that x^{I} processes reaching by the time instant t' the domain $\overline{G}_\delta$ with the second component of V solution satisfying $\mu(t') = -\operatorname{sgn}\sigma(x^{\mathrm{I}}(t'))$, subject to

$$k_i^0 > \overline{\overline{k}}_i^0, \quad i = 1, 2 \tag{6.135}$$

appear in G_0 in a finite time not exceeding

$$\Delta t = \max_{i=1, 2;\ \widetilde{c} \in \Sigma_2} \delta\beta_i / \widetilde{c}\boldsymbol{b}\,(k_i^0 - \overline{\overline{k}}_i^0) \tag{6.136}$$

and later the bound (6.126) is ensured for these processes. These reasonings lead us to define the conditions for the $\mathcal{P}_x$-process being $\mathcal{A}_\mu(2)$ controlled, i.e. for the convergence of all controlled processes in the $S_\mu(2)$ system (6.55), (6.122) to the point O_x. Below we display the theorem accurately formulating these conditions.

Theorem 6.11-4. Let relationships (6.125), (6.128), (6.130), and (6.135) hold for the $S_\mu(2)$ system (6.55), (6.122) at some positive constants c and δ. Then all controlled processes $\boldsymbol{x}(t)$ in the $S_\mu(2)$ system possess the property $\lim_{t\to\infty} \|\boldsymbol{x}(t)\| = 0$ and divide themselves into two families, $\boldsymbol{x}^{\mathrm{I}}(t)$ and $\boldsymbol{x}^{\mathrm{II}}(t)$ processes. Beginning from a certain time instant, for the $\boldsymbol{x}^{\mathrm{I}}(t)$ processes the bound (6.126) holds.

What remains to prove to complete our qualitative study of the $S_\mu(2)$ system (6.55), (6.122) is that the $\mathcal{A}_\mu(2)$ control of $\mathcal{P}_x$-process is stable. The scheme of reasoning would be the same as has been used in Section 6.7 for proving a similar assertion for the $\mathcal{A}_\mu(1)$ algorithm with integral COFB. To save space, we drop it, confining ourselves to stating that for the $S_\mu(2)$ system at hand, under the conditions of Theorem 6.22-4, the bound (6.74) is valid upon substitution of $\overline{\delta}$, defined in (6.127), for δ and using for P the expression

$$P = \max_{i=1, 2;\ |d|\leqslant 1} \sup_{t\geqslant t_0} \|\boldsymbol{a}^i(t) - d\boldsymbol{b}k_i^0\| \tag{6.137}$$

where $\|\cdot\|$ stands for the octahedral norm.

We have proved thus that the $\mathcal{P}_x$-process is stable, and $\mathcal{A}_\mu(2)$ controlled by the control algorithm (6.116), (6.117), (6.55) without any additional assumptions on the $\mathcal{P}_x$-process behavior. It follows from the fact that all the conditions of Theorem 6.11-1 to 6.11-4 can be satisfied by a suitable choice of system's circuit parameters. Therefore in this case

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_\mu(2)}\left\{\mathcal{F}_c,\ I_2;\ \frac{c_m}{\overline{\delta}}\left(\frac{1+\overline{\delta}+c}{c}\right)^2 \exp\frac{2P}{\alpha}\right\} \tag{6.138}$$

where the constant $\overline{\delta}$ is given in (6.127) and P in (6.137).

All the remarks made in Section 6.7 on the dependence of the controlled process dynamics on $\mathcal{P}_x$ parameters and $\mathcal{A}_\mu$ (2) parameters apply here as well. It will be noted that similar results take place for $\mathcal{A}_\mu$ (2) algorithms with other COFB laws, as those described in Sections 6.8-6.10. Figure 6.15 provides a correct insight into the character of motions in such S_μ (2) systems.

Summarizing the investigation of this section we are in a position to say that the $\mathcal{A}_\mu$ (2) control algorithms do not introduce principal alterations into the controlled process behavior as compared to those achievable with $\mathcal{A}_\mu$ (1) algorithms. It should, however, be recognized that this case is advantageous by a more convenient procedure for computing S_μ (2) system parameters, attainable at the expense of certain sophistication of the control algorithm. More significant advantages of S_μ (2) systems concern a wider class of $\mathcal{P}_x$-processes amenable to control and, generally speaking, lower values of gains in the CFB loop.

6.12 Application of $\mathcal{A}_\mu$ Algorithms with Inertial Coordinate-Operator Feedback to S System Control

In this section we shall be concerned with adaptation of the approach to control developed in this study to dynamic plants of more general form than those treated in earlier sections. This adaptation reveals a number of salient differences due to the methods of analysis and synthesis of control systems and the properties which may be ensured for the closed-loop control system in this case. All considerations will be outlined for the case of an $\mathcal{A}_\mu$ control algorithm with inertial COFB. Of course, all results established here can be developed for other types of $\mathcal{A}_\mu$ algorithm as well. This can be readily achieved on the lines of reasoning used to advantage in the foregoing sections. The suggested analytical approach is general in nature and may be employed for controlled plants with arbitrary state space dimensionality.

Let us define the class of controlled plants to be considered below. Whereas in earlier sections we handled $\mathcal{P}_x$-processes, i.e. those described by one differential equation, this time we shall be concerned with more general type of processes described by a set of differential equations, say of first order. In Chapter 4 these processes were termed S systems. In the absence of external coordinate or additive disturbances, the S systems to be considered are defined by

$$\dot{x}_1(t) = a_{11}(t)\, x_1(t) + a_{12}(t)\, x_2(t) + b_1(t)\, u(t)$$

$$\dot{x}_2(t) = a_{21}(t)\, x_1(t) + a_{22}(t)\, x_2(t) + b_2(t)\, u(t), \quad t \geqslant t_0 \qquad (6.139)$$

where $a_{ij}(t)$, $i, j = 1, 2$, and $b_j(t)$, $j = 1, 2$ are in general some unknown scalar time functions. Further we assume these functions being bounded, piecewise continuous, and for $t \geqslant t_0$ obey

$$a_{ij}^{-} \leqslant a_{ij}(t) \leqslant a_{ij}^{+}, \quad i, j = 1, 2$$
$$b_j^{-} \leqslant b_j(t) \leqslant b_j^{+}, \quad j = 1, 2 \qquad (6.140)$$

where the constants $a_{ij}^{\pm}$ and $b_j^{\pm}$, defining the variation ranges for S system parameters, are deemed known. In equations (6.139), $x_i(t)$, $i = 1, 2$, are the S system's state variables (error or actuating signals) which, upon stipulating a control action $u(t)$ form the vector function $\boldsymbol{x}(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$, termed the *controlled process* in this context. The problem at hand consists thus in selecting a control $u(x, t)$ — this procedure is referred to as system synthesis — from the information available on the S system, viz. $I_2 = \{\boldsymbol{x}(t), a_{ij}^{\pm}, i, j = 1, 2; b_j^{\pm}, j = 1, 2; t \geqslant t_0\}$, such that the resulted closed-loop control system, S_c, so long as the form of $u(x, t)$ is not elaborated, possesses one of the properties outlined in (4.3), (4.5), (4.7), or (4.8).

In what follows we shall use the system of scalar differential equations (6.139) in vector representation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\, \boldsymbol{x}(t) + \boldsymbol{b}(t)\, u(t), \quad t \geqslant t_0 \qquad (6.141)$$

where $\boldsymbol{b}(t) = [b_1(t), b_2(t)]^{\mathrm{T}}$ is a vector column, and $\boldsymbol{A}(t) = [\boldsymbol{a}^1(t), \boldsymbol{a}^2(t)]$ is the 2×2 matrix constituted by the vector columns $\boldsymbol{a}^1(t) = [a_{11}(t), a_{21}(t)]^{\mathrm{T}}$, $\boldsymbol{a}^2(t) = [a_{12}(t), a_{22}(t)]^{\mathrm{T}}$.

It will be in order to note here that under certain assumption the system of differential equations (6.139) can be transformed to one differential equation by incorporating new variables (see, e.g. Refs. 34, 38), that is, the control of the S system reduces to that of the $\mathcal{P}_x$-process. But even when such a transformation is feasible—this is far not always the case—the resultant $\mathcal{P}_x$-process will be described by a differential equation whose parameters may not meet the assumptions deemed to be satisfied in this evaluation. Therefore, in general, the findings of foregoing sections would not be valid to apply without appropriate assumptions and restraints made to this effect.

It will be interesting to apply the approach developed in this work directly to the control of S systems specified in the form (6.139) or (6.141). In what follows we undertake this problem and demonstrate that this handling of S systems yields new features in addition to those established in preceding sections. These are connected with the convergence of controlled processes $\boldsymbol{x}(t)$ to the origin of the phase space and with the dependence of their dynamics on the S_c system parameters.

Let us define more accurately the objectives of our evaluation and the class of controlled S systems (6.141) to be handled. Following rigorously along the lines of the approach developed in this volume we define for the controlled processes $\boldsymbol{x}(t)$ of the S_c system a functional relationship $\sigma(\boldsymbol{x}(t)) = \boldsymbol{c}\boldsymbol{x}(t)$, where $\boldsymbol{c} = [c', 1]$ is a row vector and c' is a constant. Notice that the vector $\boldsymbol{c}$ differs from the earlier used vector $\boldsymbol{c} = [1, c]$ in that now c' is not necessarily positive. Of course, the form of stipulating $\boldsymbol{c}$ is not of principal importance, but the new $\boldsymbol{c}$ will be more convenient for the working of this section.

The principal objective of control, formulated earlier as $\lim_{t\to\infty} \| \boldsymbol{x}(t) \| = 0$ for all the controlled processes concerned will not be achieved in a direct way, rather we shall strive at it indirectly by creating conditions under which the equality $\sigma(\boldsymbol{x}(t)) = 0$ can be fulfilled exactly or approximately from a certain time. Then, as will be demonstrated below, under some assumptions additional to (6.140) the control problem can be solved asymptotically.

For the strict equality of $\sigma(\boldsymbol{x}(t)) = 0$ on the S_c system solutions beginning from some time instant t we speak of the required dependence of controlled process dynamics on the parametric disturbances ($\boldsymbol{A}(t)$ and $\boldsymbol{b}(t)$) acting upon the S system. When this equality is fulfilled only approximately, say in the sense of $|\sigma(\boldsymbol{x}(t))| \leqslant \delta \| \boldsymbol{x}(t) \|$ satisfied for some sufficiently small constant $\delta > 0$ and time posterior to some instant t, we speak of a small deviation of controlled process dynamics from the specified dependence. Whenever the constant δ can be selected at will, the control law parameters of course have to be altered to this effect, we say about the possibility to change the dependence of controlled process dynamics on S_c system parameters in the desired direction. Further all these concepts will be elucidated by invoking some differential equations.

The concepts just introduced differ from similar concepts used in foregoing sections in the context of $\mathcal{P}_x$-process control. Whereas earlier we spoke of the dynamic properties of controlled processes being independent of $\mathcal{P}_x$ parameters (referred to as being "parametrically astatic"), now in S system control we in general may speak only of the controlled process dynamics as having a required dependence on S system parameters or a dependence which deviates only insignificantly from that specified in advance. This is one of the features facing the control engineer in handling S systems.

The aforestated objectives of control are attainable for S system provided some assumptions are made. Here we dwell on some of them.

Let $\boldsymbol{c} = [c', 1]$ be some fixed row vector. We construct $\tilde{\boldsymbol{c}} = \boldsymbol{c} + \xi$ with $\xi = [\xi_1, \xi_2]$ such that $\xi_i = \pm\delta$, $i = 1, 2$, and δ is some positive number. Now we formulate a proposition concerning the S system.

Proposition 6.12-1. There exist a constant vector $\boldsymbol{c}$ and $\bar{\delta} > 0$

such that for any fixed $\delta \leqslant \bar{\delta}$ the product $\tilde{\boldsymbol{cb}}(t)$ is positive for all $t \geqslant t_0$, moreover

$$\tilde{\boldsymbol{cb}}(t) \geqslant c_b = \text{constant} > 0 \tag{6.142}$$

Note that Proposition 6.12-1 is intimately connected with the controllability of the S system (6.141).

Let us establish a differential relationship between the functions $\sigma(t) = \sigma(\boldsymbol{x}(t))$ and $x_1(t)$. Given Proposition 6.12-1 holds true, compute $\dot{\sigma}(\boldsymbol{x}(t))$. Omitting the argument t throughout we get

$$\dot{\sigma} = \boldsymbol{cAx} + \boldsymbol{cbu}$$

Solving this equation for u, expressing the component x_2 of $\boldsymbol{x}$ from $\sigma(\boldsymbol{x}) = c'x_1 + x_2$ as $x_2 = \sigma - c'x_1$, and substituting the resultant expressions into the first equation of (6.139) we obtain the desired differential relationship between $x_1(t)$ and $\sigma(t)$ (referred to as differential constraint) as

$$\dot{x}_1(t) = a_c(t)\, x_1(t) + b_c(t)\, \sigma(t) + b_c'(t)\, \dot{\sigma}(t), \quad t \geqslant t_0 \tag{6.143}$$

where the functions $a_c(t)$, $b_c(t)$, and $b_c'(t)$ are related to the initial parametric functions of the S system as

$$\begin{aligned} a_c &= a_{11} - b_c' \boldsymbol{ca}^1 - c' b_c \\ q_c' &= b_1 / \boldsymbol{cb} \\ b_c &= a_{12} - b_c' \boldsymbol{ca}^2 \end{aligned} \tag{6.144}$$

For S systems, the differential relationship (6.143) plays the role of $\dot{x} = -x/c - \sigma/c$ employed in the evaluation of $\mathcal{P}_x$-process control systems. The parameters in (6.143) are unknown and time variable, their ranges of variation, however, are known and can be determined by (6.144) and inequalities (6.140). This is another feature that manifests itself in handling parametrically ill-defined S systems. We shall exploit the notation introduced above to formulate some more propositions concerning the S system.

Proposition 6.12-2. There exist a constant vector $\boldsymbol{c} = [c', 1]$ and a constant $\delta_0 > 0$ such that for all $t \geqslant t_0$

$$a_c(t) \leqslant -\delta_0 \tag{6.145}$$

Again we confine ourselves to notice only that if the S system is controlled, then Proposition 6.12-2 can be satisfied by a suitable choice of vector $\boldsymbol{c}$.

The following proposition constrains the family of vectors $\boldsymbol{b}(t)$ involved in the S system description such as (6.139).

Proposition 6.12-3. For any constant vector $\boldsymbol{c}$ validating Propositions 6.12-1 and 6.12.-2, and positive constant b_c^0

$$| \dot{b}_c' (t) | \leqslant b_c^0 \tag{6.146}$$

for all $t \geqslant t_0$.

The foregoing propositions provide an accurate characterization for the class of controlled S systems to be dealt with below. For convenience of further reference to this class we give the following definition.

Definition 6.12-1. We write $S \in \mathscr{S}(\delta_0)$ and say that the S system (6.141) belongs to the class $\mathscr{S}(\delta_0)$ if Propositions 6.12-1 through 6.12-3 are valid simultaneously.

It should be noted once again that this work does not touch upon how the property $S \in \mathscr{S}(\delta_0)$ is related to such fundamental properties of the controlled S system as, for example, controllability or stability. We confine ourselves to mentioning that these properties of the S system are intimately connected with the property $S \in \mathscr{S}(\delta_0)$.

Now we concentrate on the relationships characterizing the $\mathscr{A}_\mu$ algorithm employed to control the S system (6.141). We have promised above to consider an $\mathscr{A}_\mu$ algorithm with inertial COFB, accordingly we choose the $\mathscr{A}_\mu(1)$ algorithm for our consideration. According to the pertinent earlier definitions, this implies that the principal control signal $u(t)$ is formed as

$$u(t) = \mu(t) k_1^0 | x_1(t) | \tag{6.147}$$

where the positive constant k_1^0 is the control system's circuit parameter, and the function $\mu(t)$ interpreted as an operator variable is defined by

$$\dot{\mu}(t) = -\alpha [\mu(t) + \operatorname{sgn} \sigma(\boldsymbol{x}(t))], \quad | \mu(t_0) | \leqslant 1, \quad t \geqslant t_0 \tag{6.148}$$

Here $\sigma(\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{x}$ and a positive constant α is another parameter of the qcontrol algorithm.

Ehuations (6.147) and (6.148) specify an $\mathscr{A}_\mu(1)$ control algorithm witi inertial COFB, and together with the S system (6.141) they defne the $S_\mu(1)$ system to be evaluated below. Obviously, the solutions to such an S_μ system will be represented by a pair of functions $(\boldsymbol{x}(t), \mu(t))$. The initial condition for $\mu(t)$, given in (6.148), singles out amidst all the $(\boldsymbol{x}(t), \mu(t))$ solutions the so-called V solutions. We shall also take up the V' solutions for which $| \mu(t_0) | < 1$. All the remarks concerning the general properties of the S_μ system's V solutions made in Section 6.8 apply here to the V solutions of the system (6.141), (6.147)-(6.148) as well.

The adopted $\mathscr{A}_\mu$ algorithm means that the domain G_δ is specified as

$$G_\delta = \{\boldsymbol{x}: | \sigma(\boldsymbol{x}) | \leqslant \delta | x_1 |, \ \sigma = \text{constant} > 0, \quad \delta \leqslant \bar{\delta}\} \tag{6.149}$$

such that if $x(t) \in G_\delta$ we say that on the solutions of the S_μ system considered the required functional relationship $\sigma(x(t)) = 0$ is fulfilled approximately. For a sufficiently small value of the constant δ, we may say in this case that the dependence of the controlled process dynamics on plant parameters is close to the given dependence. Therefore, as in the foregoing sections, our investigation will be focused on the evaluation of conditions under which the domain G_δ is attracting (Definition 6.7-1) and G_0 invariant (Definition 6.7-2). Accordingly, the investigation scheme for the S_μ system (6.141), (6.147)-(6.148) coincides in general with the one outlined in Section 6.7 as consisting in solving problems A through E.

In our first evaluation we shall be concerned with the convergence to the origin of the phase space for those controlled processes of the S_μ system (6.141), (6.147)-(6.148) which beginning from a certain time belong to G_δ. This evaluation makes up the subject matter of Problem A.

Problem A. The behavior of the controlled processes $x(t) \in G_\delta$ is investigated with the differential relationship (6.143) and the inequality $|\sigma| \leqslant \delta |x_1|$. Rigorous conditions for the convergence of such controlled processes to O_x are given by the following theorem and its corollary.

Theorem 6.12-1. Let $S \in \mathcal{S}(\delta_0)$, for some V solution of the S_μ system (6.141), (6.147)-(6.148) $x(t) \in G_\delta$ for t in $[t_1, t_2]$, $t_2 > t_1 \geqslant t_0$, and

$$\delta \sup_{t \geqslant t_0} |b_c'(t)| < 1 \tag{6.150}$$

Then for this V solution on the interval $[t_1, t_2]$ we have the estimate

$$\|x(t)\| \leqslant (1 + |c'| + \delta) \frac{1 + \delta \sup\limits_{t \geqslant t_0} |b_c'(t)|}{1 - \delta \sup\limits_{t \geqslant t_0} |b_c'(t)|} \times \|x(t_1)\| \exp\{-(\delta_0 - \delta M)(t - t_1)\} \tag{6.151}$$

where the constant M is defined as

$$M = \frac{\sup\limits_{t \geqslant t_0} |b_c(t) + a_c(t) b_c'(t) - \dot{b}_c'(t)|}{1 - \delta \sup\limits_{t \geqslant t_0} |b_c'(t)|} \tag{6.152}$$

Similar to the like theorems and corollaries formulated above, this theorem and its corollary mean that given the conditions (6.150), or the like, and the inequality $|\sigma(t)| \leqslant \delta |x_1(t)|$ are met, the exponential stability of the elementary solution to $\dot{x}_1(t) = a_c(t) x_1(t)$ is not violated in the presence of a consistent disturbance $b_c(t) \sigma(t) +$

$b_c'(t)\,\dot{\sigma}\,(t)$. We display the proof of this theorem in full as it contains new elements of reasoning as compared with that for Theorem 6.7-1.

Proof. For $t \geqslant t_1$ consider a V solution to the S_μ system (6.141), (6.147)-(6.148) satisfying the conditions of the theorem. This solution satisfies (6.143) and $|\sigma(x(t))| \leqslant \delta |x_1(t)|$ for $t \geqslant t_1$. Let us represent the solution to equation (6.143) in integral form treating the function $b_c(t)\,\sigma(t) + b_c'(t)\,\dot{\sigma}(t)$ as a disturbance, namely,

$$x_1(t) = x_1(t_1) \exp\left\{\int_{t_1}^{t} a_c(\gamma)\,d\gamma\right\} + \int_{t_1}^{t} \exp\left\{\int_{\tau}^{t} a_c(\gamma)\,d\gamma\right\} \times [b_c(\tau)\,\sigma(\tau) + b_c'(\tau)\,\dot{\sigma}(\tau)]\,d\tau$$

Integrating by parts carries this expression to

$$\begin{aligned} x_1(t) = {} & x_1(t_1) \exp\left\{\int_{t_1}^{t} a_c(\gamma)\,d\gamma\right\} \\ & + \left(+ \int_{t_1}^{t} \exp\left\{\int_{\tau}^{t} a_c(\gamma)\,d\gamma\right\} b_c(\tau)\,\sigma(\tau)\,d\tau \right. \\ & + b_c'(t)\,\sigma(t) - b_c'(t_1)\,\sigma(t_1) \exp\left[\int_{t_1}^{t} a_c(\gamma)\,d\gamma\right] \\ & - \int_{t_1}^{t} \exp\left\{\int_{\tau}^{t} a_c(\gamma)\,d\gamma\right\} [\dot{b}_c'(\tau) - a_c(\tau)\,b_c'(\tau)]\,\sigma(\tau)\,d\tau \end{aligned} \tag{6.153}$$

Estimating the expressions on the right-hand side by absolute value and observing that $S \in \mathcal{S}(\delta_0)$ and $|\sigma(t)| \leqslant \delta |x_1(t)|$ by the theorem conditions, we arrive at the integral inequality

$$\begin{aligned} |x_1(t)| \leqslant {} & [1 + \delta |b_c'(t_1)|\,]\,|x_1(t_1)| \exp\{-\delta_0(t - t_1)\} \\ & + \delta |b_c'(t)|\,|x(t_1)| + \delta \int_{t_1}^{t} |b_c(\tau) + a_c(\tau)\,b_c'(\tau) - \dot{b}_c'(\tau)| \\ & \times \exp\{-\delta_0(t - \tau)\}\,|x_1(\tau)|\,d\tau \end{aligned} \tag{6.154}$$

Collecting like terms, multiplying the result by $\exp[+\delta_0(t - t_1)]$, invoking the Gronwall-Bellman lemma, and multiplying the resultant inequality by $\exp[-\delta_0(t - t_1)]$ we get, with the notation of (6.152),

$$|x_1(t)| \leqslant \frac{1 + \delta \sup\limits_{t \geqslant t_0} |b_c'|}{1 - \delta \sup\limits_{t \geqslant t_0} |b_c'|}\,|x_1(t_1)| \exp\{-(\delta_0 - \delta M)(t - t_1)\} \tag{6.155}$$

In order to derive the norm estimate for the controlled processes—recall that throughout this work we employ the octahedral norm $\|x\| = |x_1| + |x_2|$ — we resort to the inequality $\|x\| \leqslant (1 + |c'| + \delta)\,|x_1|$ which follows from the sequence

$$\|x\| = |x_1| + |x_2| = |x_1| + |\sigma - c'x_1| \leqslant |x_1| + \delta\,|x_1| + |c'|\,|x_1|$$

Substituting the bound (6.155) in the above inequality and observing that always $| x_1 | \leqslant \| x \|$ we immediately arrive at the bound (6.151) thus proving the theorem.

With the bound (6.155) on hand it is an easy matter to formulate the conditions for the controlled processes $x(t) \in G_\delta$ to converge to the origin of the phase space. These conditions are given by the following statement.

Corollary 6.12-1. Let all the conditions of Theorem 6.12-1 be satisfied and

$$\delta M < \delta_0 \tag{6.156}$$

then for any controlled process of the S_μ system (6.141), (6.147)-(6.148) such that $x(t) \in G_\delta$ for $t \geqslant t_1$ we have $\| x(t) \| \to 0$ as $t \to \infty$.

In other words, when $S \in \mathcal{S}(\delta_0)$ a value of parameter δ can be readily indicated such that all the controlled processes in the S_μ system (6.141), (6.147)-(6.148) converge to O_x. It is important therefore to reveal the conditions under which $x(t) \in G_\delta$. They will be evaluated below.

Problem B. Here we evaluate the conditions under which the domain G_δ is G_0 invariant for the S_μ system at hand. For convenience of formulating such conditions we introduce the following constants and sets:

$$\Sigma_1' = \{\tilde{c} : \tilde{c} = [\tilde{c}', 1],\ \tilde{c}' = c' + \xi,\ \xi = \pm\delta\} \tag{6.157a}$$

$$\Sigma_2' = \{\tilde{c} : \tilde{c} = [\tilde{c}', 1],\ \tilde{c}' = c' + \lambda\xi,\ |\lambda| \leqslant 1,\ \xi = \pm\delta\} \tag{6.157b}$$

where, as before, c' is the first component of the constant vector $c = [c', 1]$,

$$\bar{k}_1^0 = \max_{\tilde{c} \in \Sigma_1'} \sup_{t \geqslant t_0} \frac{|\tilde{c}a^1(t) - \tilde{c}a^2(t)\tilde{c}'|}{|\tilde{c}b(t)|} \tag{6.158}$$

$$\tilde{k}_1^0 = \sup_{t \geqslant t_0} \frac{|ca^1(t) - ca^2(t)c'|}{|\tilde{c}b(t)|} \tag{6.159}$$

$$\bar{\bar{k}}_1^0 = \max_{\tilde{c} \in \Sigma_2'} \sup_{t \geqslant t_0} \frac{|\tilde{c}a^1(t) - \tilde{c}a^2(t)\tilde{c}'|}{|\tilde{c}b(t)|} \tag{6.160}$$

where the sets Σ_1' and Σ_2' are displayed in (6.157) and the vector functions $a^i(t)$, $i = 1, 2$, and $b(t)$ are borrowed from (6.140) and (6.141).

It is not hard to verify that the values, or at least estimates, for the aforelisted constants can be found for the class of S systems under consideration.

The following theorem defines the condition for G_δ to be G_0 invariant.

Theorem 6.12-2. Let h be some fixed positive number between zero and unity and for the S_μ system (6.141), (6.147)-(6.148) Proposition 6.12-1 be satisfied and

$$k_1^0 > \bar{k}_1^0/(1-h) \tag{6.161}$$

$$\alpha \geqslant \max_{\tilde{c}\in\Sigma_2'} \sup_{t\geqslant t_0} \frac{\tilde{c}b(t)}{\delta}(k_1^0+\bar{\bar{k}}_1^0)\ln\frac{2}{h} \tag{6.162}$$

Then the domain G_δ is G_0 invariant for such an S_μ system.

It will be noted that relationships (6.161) and (6.162) are very much alike to (6.85) and (6.87), the distinction being only in the values of constants $\bar{k}_1^0$ and $\bar{\bar{k}}_1^0$. Therefore, in this case, all the remarks given above in the discussion of Theorem 6.8-1 apply as well. Moreover, the proof of Theorem 6.12-2 is similar to that for Theorem 6.8-1, so that we confine ourselves to giving here only brief interpretations.

Interpretation. Since this theorem, when proved in detail, incorporates some elements of the reasoning outlined in proving Theorems 6.8-1 and 6.11-2, we limit ourselves here to deriving the expression for the function $\lambda(t) = \sigma(\boldsymbol{x}(t))/\delta \mid x_1(t) \mid$ and the following inequalities which enable us to establish that $-\mu(t')\operatorname{sgn}\sigma(\boldsymbol{x}(t')) \geqslant 1-h$ for that V solution to the S_μ system on hand which satisfies all the conditions of Theorem 6.12-2, i.e., $\boldsymbol{x}(t_0) \in G_0$ and there exists a time instant t' such that $\boldsymbol{x}(t') \in \partial G_\delta$, when $\mid \lambda(t') \mid = 1$, where ∂G_δ is the boundary of G_δ.

In a manner similar to that used with (6.131) we have for $t \geqslant t_1 \geqslant t_0$

$$\begin{aligned}\lambda(t) &= \lambda(t_1) + \int_{t_1}^{t} \frac{\dot{\sigma}(\tau) - \lambda(\tau)\,\delta\,|\,x_1(\tau)|^{\bullet}}{\sigma\,|x_1(\tau)|}\,d\tau \\ &= \lambda(t_1) + \int_{t_1}^{t} \frac{\tilde{c}a^1x_1 + \tilde{c}a^2x_2 + \tilde{c}b\mu k_1^0|x_1|}{\delta|x_1|}\,d\tau\end{aligned} \tag{6.163}$$

where the following shorthand notation is used $\tilde{\boldsymbol{c}}(t) = \boldsymbol{c} - \delta\lambda(t)$ $S_1^x(\tilde{t})$ is a row vector, and S_1^x the row vector with elements $s_1(t) = \operatorname{sgn} x_1(t)$ and $s_2(t) = 0$.

It is assumed that $\boldsymbol{x}(t) \in G_0$ at $t = t_1$, i.e., $\lambda(t_1) = 0$, and $\lambda(t)$ retains its sign on the interval $(t_1, t']$. The variable $x_2(t)$ can be represented as $x_2(t) = \sigma(t) - c'x_1(t) = \delta\lambda(t) \mid x_1(t) \mid - c'x_1(t) = -\tilde{c}'x_1(t)$, where $\tilde{c}' = c' - \delta\lambda \operatorname{sgn} x_1$. Vector $\tilde{\boldsymbol{c}}$ and scalar $\tilde{c}'$ are related as $\tilde{\boldsymbol{c}} = [\tilde{c}', 1]$.

Incorporating $x_2(t) = -\tilde{c}' x_1(t)$ into the integrand of (6.163) carries it, after some manipulations, to

$$\lambda(t) = \lambda(t_1) + \int_{t_1}^{t} \frac{(\tilde{c}a^1 - \tilde{c}a^2\tilde{c}')x_1 + \tilde{c}b\mu k_1^0 |x_1|}{\delta |x_1|} d\tau \qquad (6.164)$$

Performing modulo estimation, this equality can be reduced to

$$|\lambda(t)| \leqslant |\lambda(t_1)| + \int_{t_1}^{t} \frac{|\tilde{c}b|}{\delta} (\bar{\bar{k}}_1^0 + k_1^0) d\tau \qquad (6.165)$$

where the constant k_1^0 is given in (6.160).

Observing in this inequality first relationship (6.162), then the estimates for the time interval $t - t_1$ (6.89), and the fact that $|\lambda(t_1)| = 1$, by assumption, leads us to the desired inequality $-\mu(t') \cdot \operatorname{sgn} \sigma(t') \geqslant 1 - h$. Observing this inequality in (6.164) with account of (6.161) indicates that the bound $|\lambda(t)| \leqslant 1$ is valid for all $t \geqslant t_1$, hence, the domain G_δ is G_0 invariant. This remark concludes our interpretations for Theorem 6.12-2.

Pausing for an intermediate account, we have proved that the relationships between the S system parameters and $\mathcal{A}_\mu$-algorithm parameters stated in Theorems 6.12-1 and 6.12-2 and Corollary 6.12-1 warrant convergence to the origin of the phase space for those controlled processes in the S_μ system (6.141), (6.147)-(6.148) which take off from G_0. Now we are to determine conditions which ensure for any controlled process either that it appears in G_0 or that $\lim_{t \to \infty} \|x(t)\| = 0$.

Prior to embarking on the search we make a remark, similar to Remarks 6.8-1, 6.9-1, and 6.10-1, concerning a refined exponential bound for the norms of controlled processes $x(t) \in G_\delta$. We shall deliver it without proof as in all similar cases above.

Remark 6.12-1. Let h be some fixed number, $0 < h < 1$, $S \in \mathcal{S}(\delta_0)$, inequalities (6.150), (6.161), and (6.162) hold, and $(1 - h) k_1^0 > \bar{\bar{k}}_1^0$. Define also $\bar{h} = 1 - \bar{\bar{k}}_1^0 / k_1^0$. Then for $t \geqslant t_1$ we have for the controlled processes considered in Theorem 6.12-2 $x(t) \in G_{\delta_h} \subset G_\delta$ and

$$\|x(t)\| \leqslant (1 + |c'| + \delta_h) \frac{(1 + \delta_h \sup\limits_{t \geqslant t_0} |b_c'|)}{(1 - \delta_h \sup\limits_{t \geqslant t_0} |b_c'|)}$$

$$\times \|x(t_1)\| \exp\left[-(\delta_0 - \delta_h M_h)(t - t_1)\right] \qquad (6.166)$$

where $\delta_h = \delta\varphi(h)$, $\varphi(h) = \ln(2/h)/\ln(2/\bar{h}) < 1$, and the expression for M_h results from (6.152) upon substituting δ_h for δ. It is quite

obvious also that once the inequality $\delta_0 > \delta_h M_h$ is satisfied, then for the said controlled processes we have $\lim_{t\to\infty} \| x(t) \| = 0$.

Having finished with the preparatory remarks we may now turn to solving the problem of attraction of the domain G_δ and establishing the fact that the control algorithm (6.147)-(6.148) ensures stable $\mathcal{A}_\mu$ control for S systems of the class $\mathcal{S}(\delta_0)$.

Problems C-E. In solving these problems we shall resort to the concepts of G_δ attraction and $\mathcal{C}$-attraction introduced by Definitions 6.7-1 and 6.7-3. In our discussion we shall follow the scheme of reasoning utilized in Section 6.8. To the S_μ system we put in correspondence the VS_h system described, for the constant h, $0 < h < 1$, by the vector differential equation, $t \geqslant t_1$,

$$\dot{x}(t) = A(t)x(t) - b(t)k_1^h(t) \mid x_1(t) \mid \operatorname{sgn} \sigma(x(t)) \tag{6.167}$$

where $\sigma(x) = cx$, and $k_1^h(t)$ is a scalar function of time satisfying for $t \geqslant t_1$

$$k_1^h(t) > k_1^0(1-h) \tag{6.168}$$

The relation of the VS system (6.167)-(6.168) to the S_μ system (6.141), (6.147)-(6.148) clarifies with the aid of the remarks similar to those made in Section 6.8 on the relation of the VS_h system (6.91)-(6.92) to the S_μ system (6.56), (6.84).

With these definitions we are in a position to formulate a theorem, which is a direct analog of Theorem 6.8-2, by replacing the VS_h system (6.91)-(6.92) there with the VS_h system (6.167)-(6.168). Recall that this theorem is to establish the general conditions of G_δ attraction for the S_μ system (6.141), (6.147)-(6.148) and reduce them to the condition of $\mathcal{O}$-attraction by the domain G_0 for the VS_h system (6.167)-(6.168). The theorem, accordingly, concerns the sufficient conditions of $\mathcal{O}$-attraction for the VS_h system (6.167)-(6.168). These conditions ensure G_δ attraction for the S_μ system (6.141), (6.147)-(6.148).

Theorem 6.12-3. Given $S \in \mathcal{S}(\delta_0)$, h some positive constant, $0 < h < 1$, and

$$\sup_{t\geqslant t_0} ca^2(t) < 0 \tag{6.169}$$

$$k_1^0(1-h) > \tilde{k}_1^0 \tag{6.170}$$

where k_1^0 is given in (6.158). Then G_0 is $\mathcal{O}$-attracting for the VS_h system (6.167)-(6.168).

We note that in handling S systems the condition (6.169) replaces (6.67) exploited in $\mathcal{P}_x$-process control, i.e. now this condition also limits the class of admissible S systems.

The proof of the theorem will be preceded by a lemma taking the

part of Lemma 6.7-1 in the proof of Theorem 6.7-4. This lemma occurs in various versions and has been proved in many treatises on control theory. These proofs, as a rule, overlook the fact that such lemmas are in turn direct corollaries of some more frequent assertions. In what follows we recognize this connection and note that it helps clarify some topics concerned with the type of differential equations describing the motion in discontinuous dynamic systems under sliding modes and also with the dependence of solutions to such differential equations upon "small deformations" on their right-hand sides. A detailed discussion of this topic would however lead us astray of the objective of this study.

Lemma 6.12-1. Let for $t \geqslant t_1$ there be given the scalar differential equation

$$\dot{x}(t) = a_c(t)\, x(t) + b_c(t)\, \chi(t) + b_c'(t)\, \dot{\chi}(t) \tag{6.171}$$

where $\chi(t)$ is a continuous function defined for $t \geqslant t_0$ and such that $|\chi(t)| \to 0$ as $t \to \infty$; $a_c(t)$, $b_c(t)$, and $b_c'(t)$ are some functions displayed in (6.144) and satisfying Propositions 6.12-1 through 6.12-3. Then for an arbitrary solution $x(t)$ to equation (6.171) $\lim_{t\to\infty} x(t) = 0$.

Proof. Like in the proof of Theorem 6.12-1, for $t \geqslant t_1$ for an arbitrary solution $x(t)$ to equation (6.171) with the initial condition $x(t_1)$ we can obtain an expression of the type (6.153) with function $\sigma(t)$ replaced there with $\chi(t)$, viz.,

$$\begin{aligned} x(t) = x(t_1) \exp\left\{\int_{t_1}^{t} a_c(\gamma)\, d\gamma\right\} + b_c'(t)\,\chi(t) \\ - b_c'(t_1)\,\chi(t_1) \exp\left\{\int_{t_1}^{t} a_c(\gamma)\, d\gamma\right\} + \int_{t_1}^{t} \exp\left\{\int_{\tau}^{t} a_c(\gamma)\, d\gamma\right\} \\ \times \left[b_c(\tau) + a_c(\tau)\, b_c'(\tau) - \dot{b}_c'(\tau)\right] \chi(\tau)\, d\tau \end{aligned} \tag{6.172}$$

Modulo estimation for the functions in (6.172) yields

$$\begin{aligned} |x(t)| \leqslant (|x(t_1)| + |b_c'(t_1)|\, |\chi(t_1)|) \exp\left\{\int_{t_1}^{t} a_c(\gamma)\, d\gamma\right\} \\ + |b_c'(t)|\, |\chi(t)| + \sup_{t\geqslant t_0} |b_c(t) + a_c(t)\, b_c'(t) - b_c'(t)| \\ \times \int_{t_1}^{t} \exp\left\{\int_{\tau}^{t} a_c(\gamma)\, d\gamma\right\} |\chi(\tau)|\, d\tau \end{aligned} \tag{6.173}$$

Given the conditions of Lemma 6.12-1 are satisfied (see the proof of Lemma 6.7-1 where the respective reasoning is elaborated), for any arbitrary small number $\varepsilon > 0$ there exists a time instant $t_1(\varepsilon)$

such that the sum of the first and third terms in (6.173) will be less than $\varepsilon/2$ for all $t \geqslant t_1(\varepsilon)$, while the second term will be less than $\varepsilon/2$ for all $t \geqslant t_2(\varepsilon)$, where $t_2(\varepsilon)$ is a time instant, in general, other than $t_1(\varepsilon)$. Therefore for all $t \geqslant \max(t_1(\varepsilon), t_2(\varepsilon))$ we have $|x(t)| < \varepsilon$, thus proving the lemma and paving the way for the proof of Theorem 6.12-3.

Proof. Consider for $t \geqslant t_1 \geqslant t_0$ an arbitrary solution $x(t)$ to the VS_h system (6.167)-(6.168) with the initial condition $x(t_1) \notin G_0$. The derivative $\dot{\sigma}(t) = c\dot{x}(t)$ along this solution has the form

$$\dot{\sigma} = ca^2\sigma + (ca^1 - ca^2c')\, x_1 - cbk_1^h |x_1| \operatorname{sgn} \sigma \qquad (6.174)$$

Given the conditions of Theorem 6.12-3 are met, equation (6.174) suggests that $\dot{\sigma}(t) < 0$ for $t \geqslant t_0$ and therefore either there exists a finite time instant t' such that $\sigma(t') = 0$, or no such instant exists but $\lim_{t\to\infty} \sigma(t) = 0$. In the last case, invoking the differential relationship (6.143) and Lemma 6.12-1 leads to $\lim_{t\to\infty} x(t) = 0$ for the considered solution, whence $\dot{x}(t) \to 0$. Because $\mathcal{O}_x \in G_0$, the domain G_0 is $\mathcal{O}$ attracting for the VS_h system (6.167)-(6.168). Q.E.D.

Now in view of the collective conditions of Theorems 6.12-1 through 6.12-3 and Corollary 6.12-1 it is not hard to prove on the lines of the reasoning used in Section 6.8, that $\mathcal{A}_\mu$-algorithms (6.147)-(6.148) render S systems (6.141) $\mathcal{A}_\mu$ controlled. This result is summarized in the following statement.

Theorem 6.12-4. Let $S \in \mathcal{S}(\delta_0)$, h be some positive constant $0 < h < 1$, and there hold expressions (6.150), (6.156), (6.162), (6.169) and the inequality

$$k_1^0(1 - h) > \bar{\bar{k}}_1^0 \qquad (6.175)$$

where $\bar{\bar{k}}_1^0$ is given in (6.160). Then all controlled processes $x(t)$ in the S_μ system (6.141), (6.147)-(6.148) possess the property of $\lim_{t\to\infty} \|x(t)\| = 0$ and divide themselves into two families—that of $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. Beginning from a certain time instant, for the x^{I} processes the estimate (6.166) is valid.

To conclude the qualitative investigation into the asymptotic behavior of controlled processes in the S_μ system (6.141), (6.147)-(6.148) it remains only to establish the validity of the estimate (4.4a) for all controlled processes for $t \geqslant t_0$. Given this is the case, then, subject to the condition (6.169), the S system is steady $\mathcal{A}_\mu$ controlled. The required estimate is provided in the following lemma, being a generalization of Lemma 6.8-1; its proof, like that of Lemma 6.8-1, is omitted.

Lemma 6.12-2. Given all the conditions of Theorem 6.12-4 are satisfied, for any controlled process $x(t)$ of the S_μ system (6.141), (6.147)-(6.148), then for $t \geqslant t_0$

$$\|x(t)\| \leqslant \frac{c_m}{\delta}(1+|c'|+\delta)^2 \frac{1+\delta \sup\limits_{t\geqslant t_0}|b_c'(t)|}{1-\delta \sup\limits_{t\geqslant t_0}|b_c'(t)|} \times \exp\{P/\alpha(\ln 2/h)\}\|x(t_0)\| \tag{6.176}$$

where $c_m = \max(1, |c'|)$, and

$$P = \max_{i=1,\ 2;\ |d|\leqslant 1} \sup_{t\geqslant t_0} \|a^i(t) - d\boldsymbol{b}(t)\, k_i^0\|$$

We summarize the results of the above investigation with an expression as follows: given $S \in \mathcal{S}(\delta_0)$ and the condition (6.169) holds true, then for the $\mathcal{A}_\mu$-algorithm (6.147)-(6.148)

$$S \in \mathcal{C}_{\mathcal{A}_\mu}\{\mathcal{F}_c,\ I;\ N\} \tag{6.177}$$

where the constant N represents the quantity preceding the norm $\|x(t_0)\|$ on the right-hand side of (6.176).

Expression (6.177) means that the parametrically ill-defined S system is steady $\mathcal{A}_\mu$ controlled in the class of continuous control actions under information I. No exponential estimate can be warranted for all controlled processes.

The general properties of binary systems for $\mathcal{P}_x$-process control are thus seen to coincide with that of S systems. The only difference consists in the nature of control process dependence on the controlled process or system parameters. This difference is disclosed below. Before we dwell on it, however, we note that the constraint (6.169) in the formulation of the property (6.177) can be lifted by changing from $\mathcal{A}_\mu$ (1) algorithms to $\mathcal{A}_\mu$ (2) algorithms (see Section 6.11).

Problem F. Here we would like to look at the dependence of controlled processes in the S_μ system (6.141), (6.147)-(6.148) on the S system parameters and circuit parameters. Turning to the differential relationship (6.143) we see that given Proposition 6.12-3 we may change the variable in this equation by

$$y(t) = x_1(t) - b_c'(t)\,\sigma(t)$$

to obtain for $t \geqslant t_0$

$$\dot{y}(t) = a_c(t)\, y(t) + [b_c(t) + a_c(t)\, \dot{b}_c'(t)]\,\sigma(t) \tag{6.178}$$

Here, as before, the function $\sigma(t)$ is treated as an external disturbance, but this time (and this is essential) the right-hand side has no component with the derivative $\dot{\sigma}(t)$. Therefore, equation (6.178)

readily suggests which differential equation can describe motion in the closed-loop control system when $\sigma(t) = \sigma(x(t)) = 0$. This equation has the form

$$\dot{y}(t) = a_c(t)\, y(t), \quad t \geqslant t_1 \tag{6.179}$$

where t_1 is a time instant from which the equality $\sigma(t) = 0$ holds identically.

We notice now that when $\sigma(t) = 0$ the equation for new variable reduces to $y(t) = x_1(t)$, and the above equation may be rewritten in the first component of controlled process, i.e. for $\sigma(t) = 0$, $t \geqslant t_1$, we have

$$\dot{x}_1(t) = a_c(t)\, x_1(t) \tag{6.180}$$

It should be recognized that in general it would not be easy to derive a similar result from equation (6.143) without invoking any additional assumptions and hypotheses. Of course, the above simple approach to deriving equations of motion over some manifold in system's state space may be extended over more general situations and types of dynamic systems.

Much interest is usually attracted towards probing equations of the type (6.179), (6.180) for insensitivity or "coarseness", i.e. for their capacity to reflect the nature of motions with the latter proceeding not exactly on a manifold, say $\sigma(x) = 0$, but rather in a narrow neighborhood. This property can readily be established with equation (6.178) and theorems on differential equations (see, e.g., Ref. 96). Such classical results suggest in particular that if the above neighborhood is characterized by the condition $\| \sigma(x) \| \leqslant \eta$, $\eta =$ constant > 0, then on a finite or infinite time interval the *solutions* to equations of the type (6.179), (6.180) are close to the exact solution; whereas for $\| \sigma(x) \| \leqslant \delta \| x \|$, $\delta =$ constant > 0, the *dynamics* of solutions of the approximating equations is close to that of the exact solutions.

The change of variable $y(t) = x_1(t) - b_c'(t)\, \sigma(t)$ reduces the proof of Theorem 6.12-1 and Lemma 6.12-1 to invoking familiar results of differential equation theory for (6.178). Thus, in proving Theorem 6.12-1 we let the function $\sigma(t)$ on the right-hand side of (6.178) satisfy the inequality

$$|\sigma(t)| \leqslant \delta\, |y(t)| / [1 - \delta \sup_{t \to \infty} |b_c'(t)|]$$

which immediately follows from $|\sigma| \leqslant \delta\, |x_1|$, provided by the conditions of Theorem 6.12-1, and $|x_1| \leqslant |y| + |b_c'|\, |\sigma|$ resulted from the expression for new variable. To prove Lemma 6.12-1, we let $\sigma(t) \to 0$ as $t \to \infty$ in equation (6.178). We have however no space to discuss this topic here at some length.

Equation (6.180) is valid whenever $\sigma(t) = 0$, but then $x_2(t) + c'x_1(t) = 0$ so that $x_2(t) = -c'x_1(t)$. This argument leads us to representing the respective controlled process as

$$\boldsymbol{x}(t) = [1, \ -c']^{\mathrm{T}} x_1(t), \quad t \geqslant t_1$$

Since c' is a constant it can be seen that for ideal motion, i.e. for $\sigma(\boldsymbol{x}(t)) = 0$, the dependence of controlled process dynamics in the S_μ system on S system parameters is completely defined by the dependence of $x_1(t)$ upon these parameters. This dependence is evaluated with equation (6.180) in turn governed completely by the function $a_c(t)$. In view of (6.144), this function can be written in terms of S system parameters as follows (the argument t is omitted throughout):

$$a_c = a_{11} - c'a_{12} + (b_1/\boldsymbol{cb})\ (c'\boldsymbol{ca}^2 - \boldsymbol{ca}^1) \tag{6.181}$$

This expression suggests that a_c depends on S system parameters, so that even in the ideal case the controlled process dynamics in S systems is dependable on S system parameters. We can only hope in such a situation to shape this dependence of a_c, and hence of $\boldsymbol{x}(t)$, on S system parameters into a desired form by a suitable choice of c', for example, so that Proposition 6.12-2 be valid for a given number δ_0. Specifically, we may try to secure the equality $\delta_0 = 1/c$, where c is a constant employed in foregoing sections to provide the desired dynamic behavior for $\mathcal{P}_x$-process control systems.

Now we look at a situation with the coefficient a_c in (6.180) being independent of the variable parameters of S system. Assume the columns $\boldsymbol{a}^1(t)$ and $\boldsymbol{a}^2(t)$ of the matrix $\boldsymbol{A}(t)$ in (6.141) and vector $\boldsymbol{b}(t)$ vary in time so that the following relations hold:

$$\boldsymbol{a}^i(t) = \boldsymbol{a}_0^i + \boldsymbol{b}(t)\,\alpha_i(t), \quad i = 1, 2 \tag{6.182}$$

$$\boldsymbol{b}(t) = \alpha_3(t)\,\boldsymbol{b}_0 \tag{6.183}$$

where $\boldsymbol{a}_0^i$, $i = 1, 2$, and $\boldsymbol{b}_0$ are constant and known vectors, and $\alpha_i(t)$, $i = 1, 2, 3$, are unknown functions of time validating the constraints (6.140) and Propositions 6.12-1 and 6.12-2. Substituting these equations into (6.181) yields

$$a_c(t) = a_c^0 = a_{11}^0 - c'a_{12}^0 + (b_1^0/\boldsymbol{cb}_0)\ (c'\boldsymbol{ca}_0^2 - \boldsymbol{ca}_0^1) \tag{6.184}$$

where zero superscripts mark the components of the respective vectors.

Now the quantity a_c^0 is constant, so providing it with a desired value, say $a_c^0 = -\delta_0$, for the given positive constant δ_0, we may resolve the resultant algebraic equation for the constant c'. If this equation has a solution, then Proposition 6.12-2 is valid and the ideal motion is independent of the variable parameters of S system and has a desired form.

Let us now look at the dependence of controlled process dynamics in the S_μ system (6.141), (6.147)-(6.148) on S system and control algorithm parameters, and also at the possibility to achieve an asymptotically independent response with respect to parametric disturbances. It should be noted at the very beginning that the dynamic properties of x^{II} processes (recall that $x^{II}(t) \notin G_\delta$ after some time instant) depend on S system parameters even with (6.182) and (6.183) being valid, and an exponential bound for the norm of such controlled processes is rather hard to derive. That is why expression (6.177) lacks the term reflecting the existence of an exponential bound for norms of all controlled processes. Therefore, here, like in $\mathcal{P}_x$-process control, we have to confine ourselves to an analysis of the dependence of x^I processes, i.e. those for which $x(t) \in G_\delta$, on plant parameters.

Following the methodology of Section 6.7, we introduce a space of parameters π_y tied with the coefficients of the differential equation characterizing the behavior of controlled processes under $x(t) \in G_\delta$. For S systems, this space would be convenient to define in the space $(y, \dot{y})$ rather than in x's as in Section 6.7. For this purpose turn to equation (6.178) rewritten as

$$\dot{y}(t) = a_c(t)\, y(t) + h_c(t)\, \sigma(t), \quad t \geqslant t_0 \tag{6.185}$$

where $h_c = b_c + a_c b_c' - \dot{b}_c'$.

To an ideal motion (see equation (6.179) in the space π_y there corresponds a nonstationary point $\pi^t = (a_c(t), 1)$, and with (6.182) and (6.183) holding true, there corresponds a fixed point $\pi^0 = (a_c^0, 1)$. Recall that the coordinates of points in this space derive from the condition $\pi \mathbf{y} = \pi [y, \dot{y}]^T = 0$, where T stands for the transpose.

With $x(t) \in G_\delta$ we have $|\sigma(t)| \leqslant \delta |x_1(t)|$ which can be reduced, as shown above, to the form $|\sigma(t)| \leqslant \bar{\delta} |y(t)|$, where

$$\bar{\delta} = \delta / [1 - \delta \sup_{t \geqslant t_0} |b_c'(t)|]$$

It is quite obvious that under these conditions there exists a bounded function $\bar{\lambda}(t)$ such that $\sigma(t) = \bar{\lambda}(t)\, \bar{\delta} y(t)$ and $|\bar{\lambda}(t)| \leqslant 1$, if $y(t) \neq 0$. Using this function we can put in correspondence to equation (6.185) in the parameter space π_y a point $\pi_t^t = (a_c(t) + h_c(t)\, \bar{\lambda}(t)\, \bar{\delta}, 1)$ whereas with expressions (6.182), (6.183) at hand, a point $\pi_t^0 = (a_c^0 + h_c^0 \bar{\lambda}(t)\, \bar{\delta}, 1)$, where h_c^0 is the upper bound for the function $h_c(t)$.

The distances Δ_{π}^{t} and Δ_{π}^{0} between points π^{t} and π_{t}^{t}, and π^{0} and π_{t}^{0} are, respectively,

$$\Delta_{\pi}^{t} = \| \pi_{t}^{t} - \pi^{t} \| = \bar{\delta} \, | h_{c}(t) \, \bar{\lambda}(t) | \tag{6.186}$$

$$\Delta_{\pi}^{0} = \| \pi_{t}^{0} - \pi^{0} \| = \bar{\delta} \, | h_{c}^{0} \, \bar{\lambda}(t) | \tag{6.187}$$

Because for the S_{μ} system concerned the condition $\bar{\lambda}(t) \to 0$ as $t \to \infty$ in general does not hold, $\mathcal{A}_{\mu}$-algorithms are incapable, even in asymptotic approach, at finite values of k^{0} and α, of providing exactly the desired dependence of controlled process dynamics on S system parameters [for (6.186)] or asymptotic independent response to parametric disturbances [for (6.187)]. What can guarantee is that the dependence of the dynamic properties of $x(t)$ on plant's parameters will be close to the desired in the sense of the inequality

$$\Delta_{\pi}^{t} = \| \pi_{t}^{t} - \pi^{t} \| \leqslant \bar{\delta} \sup_{t \geqslant t_{0}} | h_{c}(t) | \tag{6.188}$$

valid in a general case, or the presence of a weak parametric dependence in a specific case of (6.182), (6.183) when

$$\Delta_{\pi}^{0} = \| \pi_{t}^{0} - \pi^{0} \| \leqslant \bar{\delta} \, | h_{c}^{0} | \tag{6.189}$$

Nonetheless, because parameter δ, and hence $\bar{\delta}$, is of the circuit parameter variety, when it decreases the system will come arbitrarily close to the aforestated properties as can be seen from (6.188) and (6.189).

With δ tending to zero in the control algorithm on hand (actually for any $\mathcal{A}_{\mu}$-algorithms) the COFB loop gain α grows to infinity as can be seen from (6.162). In the limit, the inertial COFB becomes a bang-bang COFB, and the S_{μ} system becomes a VS system for which under the relationships and assumptions stated in Theorems 6.12-1 through 6.12-3 a portion of solutions appears in the domain G_{0} in a finite time. On this domain there occurs a sliding mode and the first component of VS system solution satisfies the differential equation (6.180). From the moment the sliding mode occurs, the dynamic properties of solutions coincide with the desired properties. This result, however, is attainable for the control actions $u(t)$ from the class $\mathcal{F}_{s}$ of time functions discontinuous at every point rather than in the class $\mathcal{F}_{c}$ of continuous time functions as required in the control problem formulation. In this respect, the properties of S_{μ} systems with $\mathcal{P}_{x}^{0}$-processes or S systems coincide, which fact is outlined by expression (6.177) coinciding in its form with expression (6.98).

Summary and Conclusions.

This chapter has been devoted to the evaluation of binary control systems with two types of feedback, namely, coordinate feedback (CFB) and coordinate-operator feedback (COFB). The behavior of

control systems has been investigated for various CFB and COFB laws providing the system with different sets of properties. The analysis has been aimed at revealing the possibility of attaining the complete solution of the control problem stated in the introductory portion of the chapter. The principal requirements to this effect have been formulated in (6.51).

It has been demonstrated that the complete solution to the control problem is unattainable with $\mathscr{A}_\mu$ control algorithms, either static or dynamic. This solution can, though, be approached, however closely, by a suitable choice of system's circuit parameters. A weak dependence of control process dynamics on plant parameters in S_μ systems may be ensured only for a portion of processes, specifically when $\mathscr{P}_x$-processes or S systems are controlled subject to (6.182) and (6.183).

For S systems of general type subject to $S \in \mathscr{S}(\delta_0)$, $\mathscr{A}_\mu$ control algorithms are capable of drawing the dependence of control process dynamics arbitrarily close to the given dependence.

For step variation of $\mathscr{P}_x$-process or S system parameters, asymptotic independence of the parameters cannot be achieved and what can be ensured in the class of admissible control actions is a weak dependence. In this context it would be interesting to investigate the capabilities of binary control systems with other types of feedback or their combination. One of such systems is fitted with operator feedback. It will be studied in the ensuing chapter.

7 Operator Feedback

In this chapter we shall be concerned with binary control systems of Chapter 6 augmented with an operator feedback loop, or OFB for short. All control systems evaluated in this chapter can be described by the generalized block diagram depicted in Fig. 7.1. In agreement with the nomenclature devised in Chapter 4, all such feedback systems will be referred to as $S_{\mu\rho}$ systems.

As can be seen from our previous discussion, $S_{\mu\rho}$ systems result from incorporating into S systems or $\mathscr{P}_x$-processes a nonlinear feedback loop with an $\mathscr{A}_{\mu\rho}$ control algorithm. We shall use a number q in the symbols for algorithms and control systems—appearing as $\mathscr{A}_{\mu\rho}(q)$ or $S_{\mu\rho}(q)$—to denote, as before, the number of components in the state vector $x(t)$ which is explicitly involved in the CFB law.

The principal object of control in this context is seen as the achievement of the complete solution to the control problem. In what follows we shall focus on the case with the operator feedback function, represented in Fig. 7.1 by the operator C_ρ, being a static one.

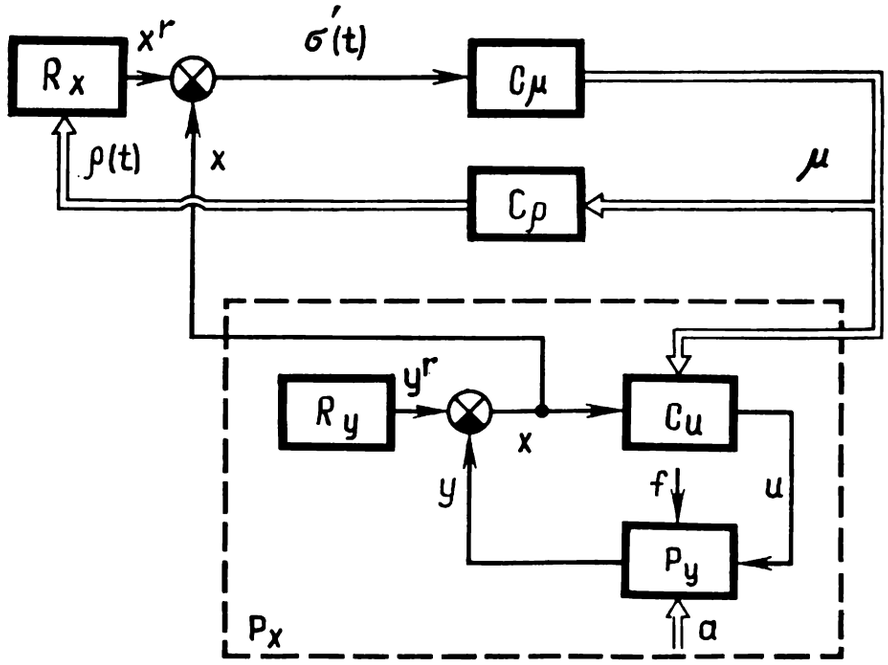

Figure 7.1

First of all we should be clear as to the objectives aimed at by introducing OFB. In Part One we mentioned that OFB is designed to ensure "nearing" of the dynamic properties of the reference (setpoint) element R_x and the S_μ system. This nearing must be achieved without any information on $\mathcal{P}_x$ parameters. Now we are going to disclose the meaning we put into the term "nearing" of dynamic properties.

In Chapter 6 we noted that S_μ control systems with dynamic COFB may experience oscillations of control processes relative to the domain $\sigma(x) = 0$. These oscillations are more clear-cut in S_μ systems with integral COFB and constant integration rate, studied in Section 6.7. Indeed, for any V solution $(x(t), \mu(t))$ to such an S_μ system we have $x(t) \in G_\delta$, subject to the conditions indicated in Section 6.7, for all $t \geqslant t_1$, where t_1 is a certain time instant. According to (6.55) and (6.63) for $|x(t)| \neq 0$ we have for such V solutions $\lambda(t) = \sigma(x(t))/\delta |x(t)|$, and the time variations for the functions $\mu(t)$ and $\lambda(t)$ are as follows:

$$\begin{aligned} \dot{\mu}(t) &= -\alpha \operatorname{sgn} \lambda(t) && \text{for } |\mu(t)| \leqslant 1 \\ &= -\omega\mu(t) && \text{for } |\mu(t)| > 1,\ |\mu(t_1)| \leqslant 1 \\ \dot{\lambda}(t) &= \Omega\mu + \Omega r(t), && t \geqslant t_1 \end{aligned} \tag{7.1}$$

where the constant $\Omega = ck^0/\delta$, $r(t) = a_1(t) - a_2(t)\,\zeta + \zeta^2$, $\zeta = [1 + \delta\xi(t)]/c$, $\xi(t) = \lambda(t) \operatorname{sgn} x(t)$, and the other designations are the same as in Chapter 6. It can be readily verified that given the domain G_δ is G_0 invariant for the S_μ system in question, $\sup_{t \geqslant t_1} |r(t)| < 1$.

If we assume that for $t \geqslant t_1$ the function $\lambda(t)$ experiences no sign reversals, then, as can be seen from the first equation of (7.1), the variable $\mu(t)$ reaches extremum in a time not exceeding $2/\alpha$, and beyond this time interval $\mu(t) = -\operatorname{sgn} \lambda(t)$. Substituting this equation into the second equation of (7.1) and accounting for the fact that $\sup_{t \geqslant t_0} |r(t)| < 1$ indicates that the function $|\lambda(t)|$ monotonously falls off and there exists a finite time instant t' when $|\lambda(t')| = 0$ and $\dot{\lambda}(t') \neq 0$. A sign reversal in $\lambda(t) = \sigma(t)/\delta |x(t)|$, however, means that the controlled processes oscillate about the domain $G_0 = \{x: \sigma(x) = 0\}$. This argument in particular suggests that the frequency of $x(t)$ oscillations about G_0 increases with α while the amplitude decreases.

If with an operator feedback the G_0 invariance of G_δ is achieved at a lower α and the aforementioned oscillations of controlled processes vanish, then we say that such OFB brings closer the dynamics of the reference input element R_x and that of the S_μ system.

There exist a number of ways to eliminate oscillations of error signal $\sigma(t)$ in the COFB loop. A traditional approach in control theory is to employ the derivative $\dot{\sigma}(t)$ in the control law—in the COFB law in this case. In this approach, for the $\mathcal{A}_\mu$-algorithm under study equation (6.55) must read

$$\begin{aligned} \dot{\mu}(t) &= -\alpha \operatorname{sgn} [\dot{\sigma}(t) + d\sigma(t)] \quad \text{for } |\mu(t)| \leqslant 1 \\ &= -\omega\mu(t) \quad \text{for } |\mu(t)| > 1, \ |\mu(t_0)| \leqslant 1, \quad t \geqslant t_0 \end{aligned} \tag{7.2}$$

where d is a positive constant characterizing "weight" of the derivative, and $\sigma(t)$ is computed on the solutions to the closed-loop control system in accord with (4.9).

The block diagram for an S_μ system with such COFB law is depicted in Fig. 7.2. To implement, such systems need a greater amount of information on the $\mathcal{P}_x$-process under control as now not only $x(t)$ and $\dot{x}(t)$ but also $\ddot{x}(t)$ must be known for $t \geqslant t_0$.

Let us have a closer look at the S_μ system with COFB law (7.2). We perform the analysis of the S_μ system (6.56), (7.2) under a number of additional assumptions and constraints, some of which occur naturally and are unavoidable, while the others are introduced for the sake of convenience. Although our discussion will be of a rather rigorous nature, no exact statement about the properties of controlled processes in such an S_μ system will be done. This is due to our prime purpose being to establish the general character of analytical findings to be employed further in comparing the properties of control systems belonging to various classes. Where necessary, a rigorous substantiation to all the results being obtained may be given on the

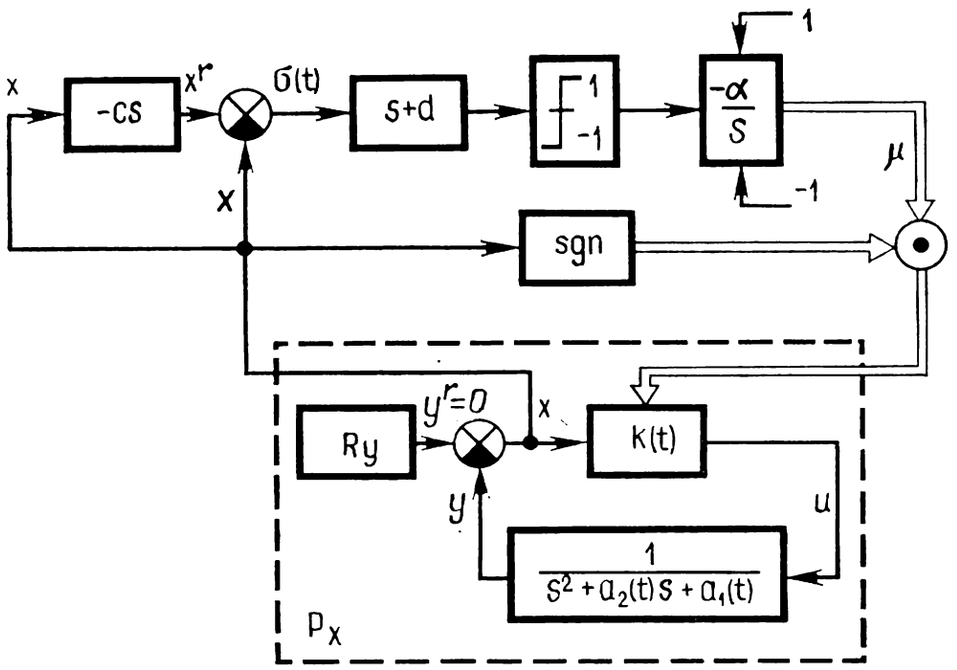

Figure 7.2

lines of the reasoning used above and partly exploited below in this section.

Like for all S_μ systems, the solutions to the S_μ system (6.56), (7.2) occur as pairs of functions (x (t), μ (t)). The conditions specified by (7.2) single out from these solutions the so-called V solutions. In what follows we limit ourselves to analyzing only those V solutions (x (t), μ (t)) for which x (t) $\in G_\delta$ as otherwise under some conditions the properties of these solutions would be close to that of the V solutions to the S_μ system (6.55)-(6.56) and their behavior then could be evaluated as in Section 6.7. This all, however, will be true provided the parameter d in (7.2) is not arbitrary but tied by a certain relationship with $\mathcal{P}_x$ parameters and control algorithm parameters. We are going to derive this relationship.

Designating

$$a_\sigma(t) = [ca_2(t) - 1]/c$$
$$a_x(t) = [ca_2(t) - 1]/c - ca_1(t) \qquad (7.3)$$

carries (6.57) to the form (t is purposefully dropped out)

$$\dot{\sigma} = -a_\sigma\sigma - a_x x + ck^0\mu \mid x \mid \qquad (7.4)$$

All further efforts will be concentrated on revealing the conditions under which in the S_μ system concerned the equality $\dot{\sigma} + d\sigma = 0$ once achieved is valid for all later times. Therefore our decision to limit the analysis to only those V solutions where $x \in G_\delta$ is correct

as from $\dot{\sigma} + d\sigma = 0$ there follows $|\sigma| \leqslant \delta |x|$. Substituting the expression for $\dot{\sigma}$ from (7.4) into this equality we get

$$\sigma = \frac{a_x - ck^0 \mu \operatorname{sgn} x}{|d - a_\sigma|} x$$

with the obvious assumption $d \neq a_\sigma$.

The required inequality will take place if d satisfies

$$\sup_{t \geqslant t_0;\ |\xi| \leqslant 1} \frac{|a_x - ck^0 \xi|}{|d - a_\sigma|} \leqslant \delta \tag{7.5}$$

Let us determine now the conditions under which we can keep identically $\dot{\sigma} + d\sigma = 0$ in the S_μ system at hand. To derive them we shall be in need of the following proposition.

Proposition 7-1. Almost everywhere on the semiaxis $t \geqslant t_0$ we have

$$\begin{aligned} |\dot{a}_\sigma| &\leqslant \text{constant} < \infty \\ |\dot{a}_x| &\leqslant \text{constant} < \infty \end{aligned} \tag{7.6}$$

These inequalities constrain the rate of $\mathcal{P}_x$ parameter variation on the continuity intervals of $a_1(t)$ and $a_2(t)$.

Finally, we make another remark on the type of the COFB law (7.2). As we have already mentioned, the value of $\omega > 0$ is immaterial in integral COFB laws, moreover for $|\mu(t)| > 1$ the very type of the equation of motion is not so significant. What is essential is that in a finite time the function $\mu(t)$ must enter the interval $[-1, 1]$. This obviously will take place, for example, when

$$\begin{aligned} \dot{\mu} &= -\alpha \operatorname{sgn} [\dot{\sigma} + d\sigma] \quad \text{for} \quad |\mu| \leqslant 1 \\ &= -\alpha \operatorname{sgn} \mu \qquad\quad \text{for} \quad |\mu| > 1 \end{aligned}$$

This expression can be rewritten in a more compact and convenient form by letting $\sigma_d = \dot{\sigma} + d\sigma$, indeed,

$$\dot{\mu} = -\alpha \operatorname{sgn} (\sigma_d + \mu |\sigma_d|), \quad |\mu(t_0)| \leqslant 1 \tag{7.7}$$

Let us obtain the derivative of $\sigma_d(t)$ for the S_μ system (6.56), (7.7) subject to $\sigma_d = 0$. After some straightforward algebra we get

$$\dot{\sigma}_d|_{\sigma_d = 0} = A_\sigma \sigma + A_x x - ck^0 \alpha |x| \operatorname{sgn} (\sigma_d + \mu |\sigma_d|) \tag{7.8}$$

where

$$A_\sigma = -\dot{a}_\sigma + a_x/c - d^2 - da_\sigma - k^0 \mu \operatorname{sgn} x \tag{7.9}$$

$$A_x = -\dot{a}_x + a_x/c - k^0 \mu \operatorname{sgn} x \tag{7.10}$$

Now we notice that given (7.5) the condition $\sigma_d = 0$ suggests that $x \in G_\delta$ and, consequently, $|\sigma| \leqslant \delta |x|$, therefore, at a sufficiently large α the sign of $\dot{\sigma}_d$ will be opposite to that of σ_d, and a sliding mode is possible on the manifold $\sigma_d = 0$ in the space (x, μ). The respective value of α is easily estimated as follows: for almost all $t \geqslant t_0$ and $|\xi| \leqslant 1$

$$\alpha > (1/ck^0) |\delta A_\sigma(\xi) + A_x(\xi)| \tag{7.11}$$

where $A_\sigma(\xi)$ and $A_x(\xi)$ outwardly coincide with A_σ and A_x, the difference concerns the substitution of ξ for $\mu \operatorname{sgn} x$ in (7.9) and (7.10). Deferring for the moment the question of whether or not there exists a finite time instant when $\sigma_d = 0$ in the S_μ system on hand, we turn to discussing the possible consequences following from $\sigma_d = 0$. In the circumstances, the COFB loop error $\sigma(t)$ varies according to an exponential law in compliance with the equation

$$\dot{\sigma} = -d\sigma, \quad d = \text{constant} > 0 \tag{7.12}$$

the exponent being independent of $\mathcal{P}_x$ parameters. This immediately suggests that no oscillations exist of the function $\sigma(t)$ about zero and, consequently, of the controlled processes, and the error $\sigma(t)$ satisfies the condition $\lim_{t\to\infty} \sigma(t) = 0$. The rate at which $\sigma(t)$ goes to zero can be set at will by an increase in d, which necessitates, however, an increase in α in the COFB law (7.7).

It is worth noting also that the neighborhood of the domain G_0 where such effect can be achieved decreases with increasing d. If the downslope of $|\sigma(t)|$ is steeper than the falling rate of the controlled processes $x \in G_\delta$, this implies that the controlled processes are getting closer to G_0, i.e., asymptotically independent behavior with respect to parameters takes place. In a formal way, this fact follows from analyzing the function $\lambda(t) = \sigma(t)/\delta |x(t)|$ representing instantaneous values of the slope for the curve passing through the origin of the phase space and a point $x(t) \in G_\delta$.

Hence the derivative of COFB loop error $\sigma(t)$ used in the COFB law does eliminate the oscillations of this error about zero and this fact is not surprising but rather it is ordinary in control theory. Another fact draws attention in this respect, namely, that in this case at least a portion of controlled processes (more specifically x^{I} processes) may behave in the limit, at bounded loop gains independently of $\mathcal{P}_x$ parameters under unknown variation laws of these parameters. For linear feedback systems with constrained gains and the same amount of information on the controlled process used in the control law, this, generally speaking, does not hold.

Still another fact that seems to be of great significance is that a negative static operator feedback in the S_μ system (6.55)-(6.56) eliminates oscillations of error signal $\sigma(t)$ about zero. Linear control

systems fail to enjoy such effect — a negative static feedback cannot cope with error oscillations.

Consider $S_{\mu\rho}$ systems resulted from augmenting S_μ systems investigated in the previous section by a static operator feedback loop in a manner depicted in the block diagram in Fig. 7.1. All control systems to be investigated below had their OFB (operator C_ρ in Fig. 7.1) generated as

$$\rho(t) = \delta\mu(t) \operatorname{sgn} x(t), \quad t \geqslant t_0 \tag{7.13}$$

where δ is the positive number involved in the description of G_δ.

With this manner of OFB organization, this feedback, like COFB, is sign-variable. We use the output variable $\rho(t)$ of the OFB loop to alter the reference input operator R_x in such a way as to get the reference variable $x^r(t)$ it generates in the form

$$x^r(t) = -c\dot{x}(t) + \rho(t)\, x(t), \quad t \geqslant t_0 \tag{7.14}$$

The associated COFB loop error $\sigma'(t) = \sigma'(x(t), \mu(t))$ then takes on the form

$$\begin{aligned}\sigma'(t) &= -\{c\dot{x}(t) + [1 - \rho(t)]\, x(t)\} \\ &= \sigma(t) + \delta\mu(t)\, |x(t)|, \quad t \geqslant t_0\end{aligned} \tag{7.15}$$

where, as before, $\sigma(t) = -c\dot{x}(t) - x(t)$.

Our further evaluation will be devoted to $S_{\mu\rho}$ systems occurring with the use of the OFB law (7.13)-(7.14) in conjunction with various COFB laws. The investigation scheme this time will be the same as was used in Chapter 6, i.e. will consist in sequential solving Problems A through F.

7.1 $S_{\mu\rho}$ (1) Systems with Integral Coordinate-Operator Feedback

The block diagram of the S_μ systems considered in this section is depicted in Fig. 7.3. The respective equations characterizing the $\mathscr{A}_{\mu\rho}$ (1) control algorithm, upon elimination of certain variables, take the form

$$u(t) = \mu(t)\, k^0\, |x(t)| \tag{7.16}$$

$$\begin{aligned}\dot{\mu}(t) &= -\alpha \operatorname{sgn} \sigma'(t) \quad \text{for} \quad |\mu(t)| \leqslant 1 \\ &= -\omega\mu(t) \quad \text{for} \quad |\mu(t)| > 1, \ |\mu(t_0)| \leqslant 1, \quad t \geqslant t_0\end{aligned} \tag{7.17}$$

where the error signal $\sigma'(t)$ is defined by (7.15).

The $\mathscr{A}_\mu$ algorithm (7.15)-(7.17) along with a $\mathscr{P}_x$-process specify the respective $S_{\mu\rho}$ system. All remarks made in Section 6.7 about

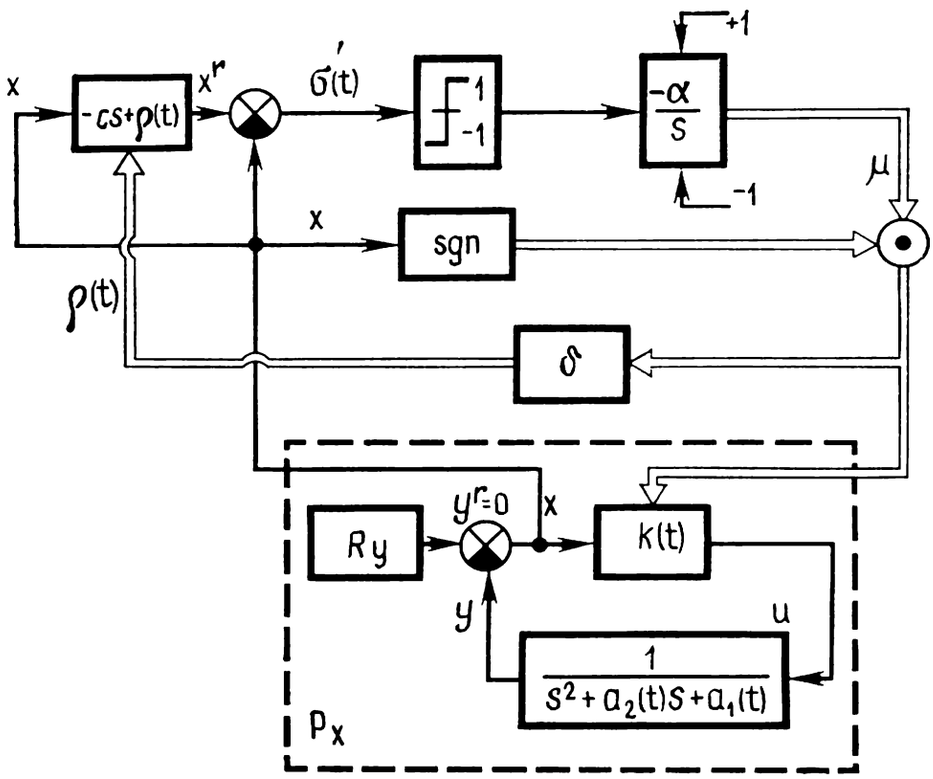

Figure 7.3

the determination of solutions $(x\ (t),\ \mu\ (t))$ to S_μ systems apply here to $S_{\mu\rho}$ systems and their general properties as well. As before, we shall limit ourselves here to evaluating only V solutions to $S_{\mu\rho}$ systems.

The asymptotic behavior of V solutions to the $S_{\mu\rho}$ system (6.56), (7.15)-(7.17) subject to $x\ (t) \in G_\delta$ is completely clarified by Theorem 6.7-1 and Corollary 6.7-1. From (7.15) it can be seen that given the inequality $|\ \mu\ (t)\ | \leqslant 1$ is satisfied — this is valid for any V solution to the $S_{\mu\rho}$ system — and the condition of $x\ (t) \notin G_\delta$, the functions $\sigma'\ (t)$ and $\sigma\ (t)$ are of the same sign. This implies that in both $S_{\mu\rho}$ and S_μ systems V solutions are described by the same differential equations. Therefore the properties of V solutions to $S_{\mu\rho}$ systems for $x\ (t) \notin G_\delta$ can be established with the respective statements from Section 6.7. This enables us to confine ourselves here to proving only one assertion similar to Theorem 6.7-3.

In the space $(x,\ \mu)$ define a domain Γ as follows:

$$\Gamma = \{(x,\ \mu):\ \sigma\ (x) + \delta\mu\ |\ x\ | = 0,\quad |\ \mu\ | \leqslant 1\} \tag{7.18}$$

This domain corresponds to the zero value of error $\sigma'\ (x,\ \mu)$ for V solutions to $S_{\mu\rho}$ system. A V solution appearing in this domain means therefore that the COFB-loop error vanishes away. The following theorem dwells on the conditions reducing the question of $\sigma'\ (t)$ vanishing to that of $\mathcal{O}$-attraction for a certain VS system.

Theorem 7.1-1. Let the domain G_0 be Θ attracting for the VS system (6.65), then the domain Γ is Θ attracting for the $S_{\mu\rho}$ system (7.15)-(7.17).

Proof. Assume the converse is true, i.e. for some V solution of the $S_{\mu\rho}$ system on the semiaxis $t \geqslant t_0$ $| \sigma (x (t)) + \delta\mu (t) | x (t) | | > 0$, moreover $\lim\limits_{t\to\infty} | (\sigma x (t)) + \delta\mu (t) | x (t) | | > 0$. These inequalities mean that for $t \geqslant t_0$ the function $\sigma' (t)$ has no sign reversals. Then examination of equation (7.17) indicates that for $t \geqslant t_0 + 2/\alpha$ we have $\mu (t) = -\text{sgn } \sigma' (t)$, so that for $t \geqslant t_0$ $\sigma (t)$ satisfies the condition $| \sigma (x (t)) | > \delta | x (t) |$, that is, $\sigma (t)$ and $\sigma' (t)$ are of the same sign. If, however, $\mu (t) = -\text{sgn } \sigma (t)$, then the controlled process $x (t)$ satisfies the VS system equation (6.65) and under the conditions of this theorem there exists a finite time instant t_2 when $\sigma (x (t_2)) = 0$ or for all t $| \sigma (x (t)) | > 0$ but with $\lim\limits_{t\to\infty} \| x \| = 0$. Both situations contradict the initial assumption, thus proving the theorem.

Hence under the conditions of Theorem 6.7-4, there exists a finite time t_1 when $\sigma' (t_1) = 0$ (these are x^{I} processes), or $\lim\limits_{t\to\infty} \| x (t) \| = 0$ and from some time instant $x (t) \notin G_\delta$ (these are x^{II} processes).

It is in order to note here also that if from some time instant $\sigma' (t) = 0$ for some V solution, then because always $| \mu (t) | \leqslant 1$, $| \sigma (t) | \leqslant \delta | x (t) |$ which is equivalent to $x (t) \in G_\delta$. Consequently, the error vanishing in the COFB loop implies that the controlled process appears in the domain G_δ. If subsequently $\sigma' (t) = 0$ then G_δ is Γ invariant. The respective conditions make up the subject matter of Problem B.

Problem B. We recall that the concept of G invariance of G_δ has been defined earlier (Definition 6.7-2) for the domain $G \subset \mathbb{R}^2_x$. Because the domain $\Gamma \subset \mathbb{R}^3_{x,\,\mu}$ the concept of Γ invariance is not defined in formal terms. We fill the gap by giving generalizing Definition 6.7-2.

Definition 7.1-1. The domain $G_\delta \subset \mathbb{R}^2_x$ will be referred to as Γ invariant ($\Gamma \in \mathbb{R}^3_{x,\,\mu}$) for S_c system, for t in $[t_1, t_2]$, $t_2 > t_1$, if for any its V solution $(x (t), \mu (t))$ such that $(x (t_1), \mu (t_1)) \in \Gamma$ we have $x (t) \in G_\delta$ for all $t \in [t_1, t_2]$. For t_2 at infinity, the time interval is not indicated.

The following theorem defines the conditions for the domain G_δ to be Γ invariant.

Theorem 7.1-2. Let for the $S_{\mu\rho}$ system (6.56), (7.15)-(7.17)

$$k^0 > \bar{k}^0 \tag{7.19}$$

$$\alpha \geqslant (c/\delta) (k^0 + \bar{\bar{k}}^0) \tag{7.20}$$

where $\bar{k}^0$ and $\bar{\bar{k}}^0$ are defined by (6.86) and (6.72), respectively. Then G_δ is Γ invariant for such an $S_{\mu\rho}$ system.

Comparing (7.19)-(7.20) with (6.60) and (6.61) for the S_μ system with integral COFB indicates that when augmented with a static OFB such an S_μ system retains the value of k^0 in the CFB loop whereas its COFB loop gain may be decreased by half. Below we present a complete proof for this theorem, for it differs significantly from those for similar results of the S_μ system.

Proof. Let $(x(t), \mu(t))$ be an arbitrary V solution to the $S_{\mu\rho}$ system (6.56), (7.15)-(7.17) satisfying the initial condition $\sigma'(t_0) = 0$.

If $x(t_0) = O_x$ then the V solution in question is $(O_x, \mu(t))$, $|\mu(t)| \leqslant 1$ for all $t \geqslant t_0$. Here $\mu(t)$ is some absolutely continuous function such that $|\dot{\mu}(t)| \leqslant \alpha$. For such a V solution, $x(t) \in G_\delta$ for $t \geqslant t_0$.

Let now $x(t_0) \neq O_x$, but then $x(t) \neq O_x$ for all $t \geqslant t_0$. Assume $t_1 \geqslant t_0$ is a time instant when $\sigma'(t_1) = 0$ but such that for all t in $(t_1, t_1 + \Delta)$ $|\sigma'(t)| > 0$. Here Δ is a positive number such that $|x(t)| > 0$ for t in $(t_1, t_1 + \Delta)$. This number has to exist as $|x(t_1)| > 0$ and $x(t)$ is continuous. At $t = t_1$ there can be the following alternative: either $|\mu(t_1)| < 1$ (case 1) or $|\mu(t_1)| = 1$ (case 2).

For case 1 we can pinpoint a positive number $\Delta_1 < \Delta$ such that for t in $(t_1, t_1 + \Delta_1)$ $|\mu(t)| < 1$, of course, at $|x(t)| > 0$ and $|\sigma'(t)| > 0$. Under these assumptions the equations for the $S_{\mu\rho}$ ysstem yield

$$\sigma'(t) = \int_{t_1}^{t} \left\{ c \left[a_1 - a_2 \left(\frac{1+\delta\xi_2}{c} \right) + \frac{(1+\delta\xi_2)(1-\delta\xi_1)}{c^2} \right] x \right.$$

$$\left. + ck^0\mu |x| - \alpha\delta |x| \operatorname{sgn} \sigma' \right\} d\tau \qquad (7.21)$$

where $\xi_1 = \mu \operatorname{sgn} x$ and $\xi_2 = \lambda \operatorname{sgn} x$.

Here, as before, we assume $|x(t)|^{\cdot} = \dot{x}(t) \operatorname{sgn} x(t)$. From equation (7.21) it is not hard to see that given (7.20) the integrand and $\sigma'(t)$ have different signs at any t in $(t_1, t_1 + \Delta_1)$. Consequently, for t close to t_1 $\sigma'(t) \equiv 0$.

Now we turn to case 2 limiting ourselves to analyzing the situation of $\mu(t_1) = 1$. Letting $\sigma'(t) > 0$ for t close to and following t_1 leads us, in view of equation (7.17), to $|\mu(t)| < 1$ for these t. This again, after invoking equality (7.21), subject to $|x(t)| > 0$, brings us to controversy meaning that $\sigma'(t) \equiv 0$ for t close to t_1.

Now we return to the main line of the proof to consider the last of remaining possibilities. Assume that $\sigma'(t) < 0$ and $|x(t)| > 0$

for t close to and following t_1, then $\mu(t) = 1$ and instead (7.21) we have

$$\sigma'(t) = c \int_{t_1}^{t} [(a_1 - a_2\zeta + \zeta^2) x + k^0 |x|] \, d\tau \qquad (7.22)$$

where $\zeta = (1 - \delta \operatorname{sgn} x)/c$.

Observing in this equation the inequality (7.19) with account of $c > 0$ we obtain that for t close to and following t_1 $\sigma'(t) > 0$ which contradicts our assumption. Hence, again for t close to t_1 $\sigma'(t) \equiv 0$. Repeating, where necessary, the above argument we convince ourselves that for $t \geqslant t_0$ the desired identity $\sigma'(t) \equiv 0$ holds true, but this identity is equivalent for V solutions to the property $x(t) \in G_\delta$ for all $t \geqslant t_0$. Q.E.D.

We have proved thereby that under the set of conditions outlined in Theorems 6.7-1, 6.7-4 and 7.1-2, and in Corollary 6.7-1 all the controlled processes in the $S_{\mu\rho}$ system (6.56), (7.15)-(7.17) converge as $t \to \infty$ to the origin O_x of the phase space. In this case we say that the $\mathcal{P}_x$-process is $\mathcal{A}_{\mu\rho}$ controlled with the control algorithm (7.15)-(7.17). These facts are accurately summarized in the following statement.

Theorem 7.1-3. Let relationships (6.59), (6.67), (7.19) and (7.20) be valid for the $S_{\mu\rho}$ system (6.56), (7.15)-(7.17) at some constants $c > 0$ and $\delta > 0$. Then the $\mathcal{P}_x$-process is $\mathcal{A}_{\mu\rho}$ controlled and all controlled processes in such an $S_{\mu\rho}$ system divide themselves into two families — that of $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. For the x^{I} processes, the bound (6.57) holds beginning from a certain time instant, while the x^{II} processes satisfy the condition of $\lim_{t \to 0} \|x^{\mathrm{II}}(t)\| = 0$ and from a certain time instant $x^{\mathrm{I}}(t) \notin G_\delta$.

It remains to prove only that the $\mathcal{P}_x$-process is steady $\mathcal{A}_{\mu\rho}$ controlled with the control algorithm on hand. The proof of this statement completely parallels that of Lemma 6.7-2 for the $\mathcal{A}_\mu$ algorithm with integral COFB, moreover, the bound (6.74) is valid in this case as well. Therefore, for the $S_{\mu\rho}$ systems (6.56), (7.15)-(7.17), subject to the assumption (6.67), we may write

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_{\mu\rho}} \left\{ \mathcal{F}_c,\ I_2;\ \frac{c_{\mathrm{in}}}{\delta} \left(\frac{1+\delta+c}{c} \right)^2 \exp(2P/\alpha) \right\} \qquad (7.23)$$

where $c_{\mathrm{in}} = \max(1, c)$, P is defined in Lemma 6.7-2, and α is determined in agreement with (7.20).

Expression (7.23) coincides with (6.82) accurate to α, therefore all remarks made in Section 6.7 for (6.82) apply here as well. Some peculiarities of control process behavior in the $S_{\mu\rho}$ system being examined will be disclosed in solving Problem F.

Problem F. Here again, as before, we confine ourselves to examining the x^{I} processes only, as the x^{II} process behavior is generally hard to draw any conclusion about except that $\lim_{t\to\infty} \| x^{\mathrm{II}}(t) \| = 0$.

A basis for our further analysis will be provided by a differential equation satisfied from some time instant by $x^{\mathrm{I}}(t)$ processes in the $S_{\mu\rho}$ system under examination.

Under the conditions of Theorem 7.1-3, any x^{I} process (henceforth we omit the superscript in the process designation) satisfies the equation $\sigma'(t) = 0$ from some finite time instant, i.e.

$$\sigma(x(t)) + \delta\mu(t) \mid x(t) \mid = 0 \tag{7.24}$$

where $\mu(t)$ is a component of the V solution corresponding to the controlled process $x(t)$.

Letting $\mid x(t) \mid > 0$ for all $t \geqslant t_1$, which is true as has been proved above provided $\mid x(t_1) \mid > 0$, we resolve (7.16) for $u(t)$ to substitute the result into (7.24). As a result for $t \geqslant t_1$ we get

$$\begin{aligned} u(t) &= -(k^0/\delta)\,\sigma(x(t)) \\ &= (k^0/\delta)\,[c\dot{x}(t) + x(t)] \end{aligned} \tag{7.25}$$

Substituting this expression for the manipulated variable in the equation for $\mathcal{P}_x$-process yields the following differential equation:

$$\ddot{x}(t) + \left[a_2(t) + \frac{k^0}{\delta}c\right]\dot{x}(t) + \left[a_1(t) + \frac{k^0}{\delta}\right]x(t) = 0, \quad t \geqslant t_1 \tag{7.26}$$

satisfied by the examined controlled processes beginning from a certain finite time (individual for each $x(t)$).

The relationship (7.24) and equation (7.25) just derived allow us to draw some qualitative conclusions on the properties of $S_{\mu\rho}$ systems and static OFB.

(*i*) The incorporation of a nonlinear static OFB into a nonlinear S_μ system (6.55)-(6.56), converts the closed-loop control system into a linear system, at least for a portion of controlled processes. We may therefore speak of a *linearizing* property of OFB.

(*ii*) For a portion of controlled processes in $S_{\mu\rho}$ systems, the manipulated variable (7.16) generated as a function of error $x(t)$ with variable gain is equivalent in its effect to a manipulated variable formed as a linear combination of the error coordinate and its first derivative with a constant gain equal to the ratio k^0/δ. Therefore we may speak of a rate property of OFB.

(*iii*) Under the assumptions made earlier, the CFB loop gain is not to exceed its maximal value k^0, but the ratio k^0/δ exceeds k^0 when the conditions of convergence are satisfied (*cf.* Corollary 6.7-1). The incorporation of an OFB thus may be said to be accompanied (again in some cases only) by the effect of *increasing CFB loop gain.*

Specifically, with $\delta \to 0$, $S_{\mu\rho}$ systems can experience the effect of infinitely large gain. But then this effect takes place in a portion of the phase space, specifically on the domain G_δ, contracting as δ decreases (as $\delta \to 0$ the domain G_δ contracts to G_0).

This is an essential difference of this indirect method of gain buildup from the direct technique when large gain may be used at any values of x (t) and $\dot{x}$ (t). This implication proves to be of decisive importance for control being exercised under deficient information on the control system status or in the presence of imperfections in the implementation of control laws, or under inaccurate modeling of the controlled process or system. Whereas in linear control systems with large gain the control problem is unsolvable altogether or has a Δ-accurate solution, binary control systems can handle it asymptotically, retaining the principal properties of the controlled processes.

It should be noted also that equation (7.26) is helpful in understanding and treating in ordinary terms the effect of a decreasing dependence of controlled processes in $S_{\mu\rho}$ systems on the $\mathcal{P}_x$ parameters a_1 (t) and a_2 (t) when δ falls off to zero. To demonstrate, in this case $k^0/\delta \to \infty$ and equation (7.26) boils down to $c\dot{x} + x = 0$ independent of a_1 (t) and a_2 (t). It should be emphasized that under the circumstances the primary control loop of the $S_{\mu\rho}$ system employs limited gains.

(iv) It is an easy matter to verify that under the conditions of Theorem 7.1-3 the solutions to equation (7.26) are aperiodic time functions. Moreover, at arbitrary fixed values of the parameters a_1 and a_2 from the range (4.2) all the controlled processes in the $S_{\mu\rho}$ system examined approach the solutions corresponding to the asymptote defined by the largest root s^* of the characteristic equation for the closed-loop control system

$$s^2 + \left(a_2 + \frac{k^0}{\delta} c\right) s + \left(a_1 + \frac{k^0}{\delta}\right) = 0 \qquad (7.27)$$

Consequently, under the aforestated conditions and constants a_i, $i = 1, 2$, the phase trajectories of the $S_{\mu\rho}$ system can evolve into a phase portrait such as that in Fig. 7.4, where the straight line $\sigma^* = 0$ represents the asymptote corresponding to the root s^*. It indicates that the aforementioned oscillations of controlled processes about the domain $\sigma(x) = 0$ with a frequency proportional to α vanish away as a static OFB is incorporated into the S_μ system. This means that under uncertain constant $\mathcal{P}_x$ parameters a static OFB is equivalent in its effect to incorporating the derivative of the error σ (t) into the COFB law. Notice that this result is achievable with a lower amount of information on the controlled process than in the case with the COFB law (7.7). An asymptotically independent

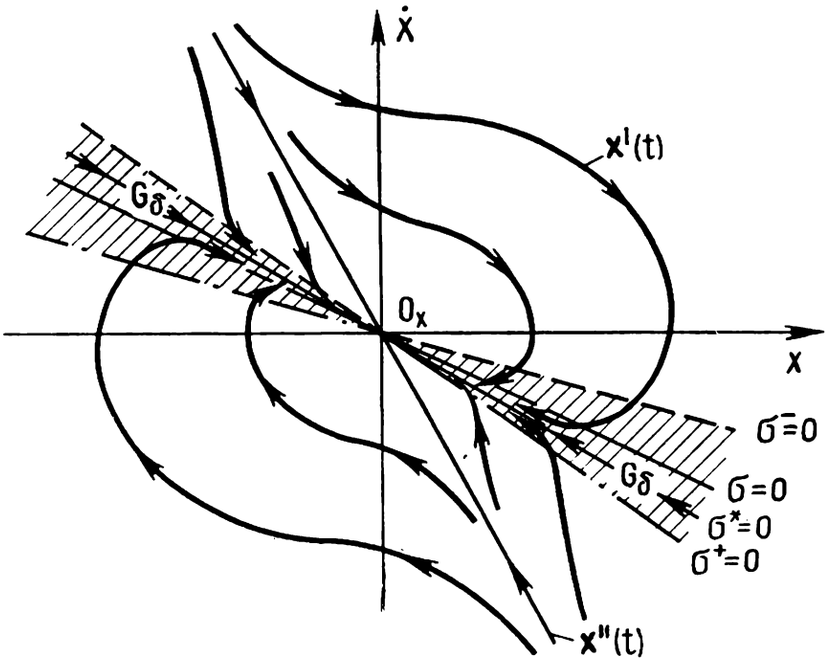

Figure 7.4

behavior with respect to parametric disturbances cannot, however, be achieved in $S_{\mu\rho}$ systems; this will be discussed at some length below.

If we now take into account this property of OFB and the fact that the COFB loop gain may be half as large in $S_{\mu\rho}$ systems as in S_μ systems, then we will be right to say that operator feedback does ensure a nearing of the dynamic properties of the reference input unit R_x and the S_μ system in the sense of the definition given at the outset of this chapter. In this respect, the objectives pursued by the incorporation of OFB are seen to be attained.

Consider now the prospects for achieving an asymptotically independent response to parametric disturbances in $S_{\mu\rho}$ systems with static OFB. For a constant parameter $\mathcal{P}_x$-process it has been established that given $x\ (t) \in G_\delta$ all controlled processes approach asymptotically the solutions of the equation $\dot{x} - s^*x = 0$, where s^* is the largest root of the algebraic equation (7.27). In the circumstances the function $\lambda\ (t) = \sigma\ (t)/\delta \mid x\ (t) \mid$ tends to the limit $\lambda^* = -[(1 + cs^*)/\delta]$ sgn x. Now we recall that for $x\ (t) \in G_\delta$ the variation of $x\ (t)$ is described by $c\dot{x}\ (t) + x\ (t) = -\delta\lambda\ (t) \mid x\ (t) \mid$ or in a more convenient form as $c\dot{x} + (1 + \delta\lambda \text{ sgn } x)\ x = 0$.

This implies that in the parameter space π to such motions corresponds a nonstationary point $\pi^t = (c,\ 1 + \delta\lambda\ (t) \text{ sgn } x\ (t))$. At fixed parameters a_1, a_2 the function $\lambda\ (t) \to \lambda^*$, therefore this point also has its limit $\pi_* = (c,\ 1 + \delta\lambda^* \text{ sgn } x) = (c,\ -cs^*)$. Since to the ideal motion (consistent with the equation $c\dot{x} + x = 0$) in the π_x space there corresponds the point $\pi_0 = (c,\ 1)$, it turns out that

under the stated conditions the distance Δ_{π}^{*} between points π_{*} and π_0 is defined as $\Delta_{\pi}^{*} = \| \pi_{*} - \pi_0 \| = | 1 + cs^{*} | = \text{constant}$. This expression means that at constant process parameters, $S_{\mu\rho}$ systems with static OFB exhibit a parametrical static (steady-state) error, or "statism" in the parlance of the Russian text, depending on the external disturbance in an ordinary manner as the value of s^{*} and, hence, the magnitude of static error Δ_{π}^{*} are decided by the values of a_1 and a_2.

Consequently, static OFB may be thought of as an ordinary static feedback. Indeed, in linear control systems, static CFB gives rise to a steady-state (static) error (or its variation) in responding to the jumps in coordinate or additive disturbances. In binary control systems, a static OFB has been recognized to exhibit a similar response but only with respect to steps in process parameter variation or with respect to multiplicative disturbances.

In the parameter space π_x a parametrical static error implies the presence of a consistent deviation of the process dynamics from the specified performance. It is essential, however, that with such a deviation no static error is exhibited by the main controlled variables and the control problem is solvable asymptotically.

Finally, we note that for $a_1(t)$ and $a_2(t)$ being functions of time, the $S_{\mu\rho}$ systems examined may, naturally, exhibit no parametrical static error. For the variable $a_1(t)$ and $a_2(t)$, only a weak dependence on parameters (parametrical "dissipatism") can be warranted since the distance between points π_0 and π^t obeys the bound

$$\Delta_{\pi}^{t} = \| \pi^t - \pi_0 \| \leqslant \delta \tag{7.28}$$

This bound suggests that both asymptotic independence and weak dependence on parameters vanish as $\delta \to 0$, but for this ultimate case the relation (7.20) indicates that α is at infinity and the $S_{\mu\rho}$ system on hand degenerates into a VS system. Accordingly, the complete solution to the control problem cannot be achieved in $S_{\mu\rho}$ systems either, but an arbitrary close approach to such solution can be ensured.

7.2 S_{μ} and $S_{\mu\rho}$ Systems Under Imperfect Information

In this section we shall be concerned with the effect of inaccuracies, occurring in deriving information on the $\mathcal{P}_x$-process being controlled, on the performance and properties of some binary control systems. The focus of our examination this time will be S_{μ} and $S_{\mu\rho}$ systems with integral COFB and static OFB; incorporation of other COFB laws results in similar conclusions.

It will be recalled that in linear control systems inaccuracies in deriving information on the derivative $\dot{x}(t)$ may render the control problem at hand insolvable under the constraints on the class of permissible controls assumed in this study. It would be of interest therefore to evaluate the behavior of binary systems under imperfect information and to compare the qualitative properties of such systems with that of linear control systems investigated in Sections 5.6-5.7.

Let us refine the problem statement. Now the $\mathcal{P}_x$-process concerned is ill-defined with the parameters in the range (4.2). The information available for control is now inaccurate and is completely defined by the set

$$\bar{I}_2 = \{x(t),\ w(t),\ t \geqslant t_0;\ a_i^{\pm},\ i = 1, 2;\ T = \text{constant} > 0\} \quad (7.29)$$

where in place of $\dot{x}(t)$, as earlier in the set I_2, there appears its bound $w(t)$ derived by actual differentiation. The differential law of $w(t)$ variation is as follows:

$$T\dot{w}(t) + w(t) = \dot{x}(t),\ t \geqslant t_0 \quad (7.30)$$

where T is some positive constant from $\bar{I}_2$. Given this information $\bar{I}_2$ for the control algorithms with integral COFB and static CFB, it is required to determine relationships between system's parameters and define the conditions under which the $\mathcal{P}_x$-process will be steady $\mathcal{A}_\mu$ and $\mathcal{A}_{\mu\rho}$ controlled.

First of all consider the block diagrams of the respective control systems. Because now the information available for control is contained in the set $\bar{I}_2$ given by (7.29), the reference input element R_x in the block diagram of Fig. 7.3 can no longer be represented in its earlier form. In forming the reference input $x^r(t)$ of the COFB loop we use the bound (7.30) in place of the ideal derivative to get

$$x^r(t) = -cw(t),\ t \geqslant t_0 \quad (7.31)$$

where c is the former constant.

This form of the function $x^r(t)$, naturally, changes the expression for the COFB loop error, now denoted by $\sigma_w(t)$, and defined under the given conditions as

$$\sigma_w(t) = -cw(t) - x(t) \quad (7.32)$$

This method of generating the reference input $x^r(t)$ and error $\sigma_w(t)$ seems to be fairly justified as for $T \to 0$ $x^r(t) \to -c\dot{x}(t)$, and $\sigma_w(t) \to \sigma(t) = -[c\dot{x}(t) + x(t)]$, so that in the limit we get the earlier examined binary control systems with ideal information.

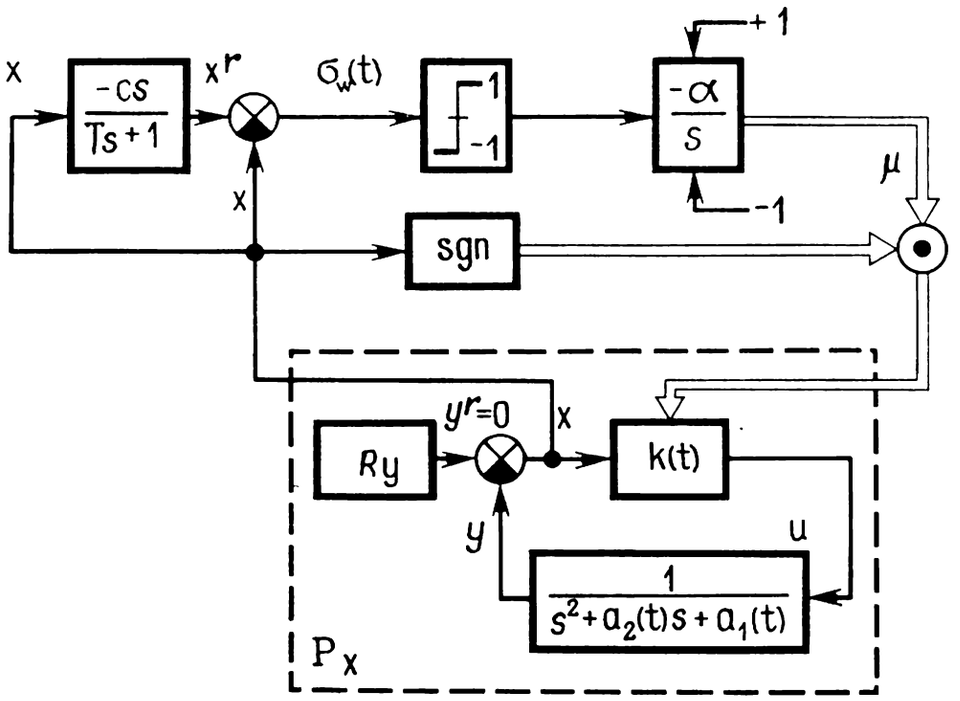

Figure 7.5

If we now invoke the integral COFB law with the operator variable $\mu(t)$ varying in accord with

$$\begin{aligned}\dot{\mu}(t) &= -\alpha \operatorname{sgn} \sigma_w(t) \text{ for } |\mu(t)| \leqslant 1 \\ &= -w\mu(t) \text{ for } |\mu(t)| > 1, \ |\mu(t_0)| \leqslant 1, \ t \geqslant t_0\end{aligned} \tag{7.33}$$

and apply the CFB law of the form (7.16), i.e.,

$$u(t) = \mu(t) k^0 |x(t)|$$

we obtain the S_μ^T system to be examined. Its block diagram is depicted in Fig. 7.5. Relationships (7.16), (7.32) and equation (7.33) specify the respective $\mathcal{A}_\mu^T$ control algorithm.

The block diagram shown in Fig. 7.6 pictures a natural way of introducing the static OFB (7.13) into the S_μ^T system on hand. We shall refer to the $S_{\mu\rho}$ system that results as the $S_{\mu\rho}^T$ system, and the respective control algorithm as the $\mathcal{A}_{\mu\rho}^T$ algorithm.

Figure 7.6 readily suggests the relationships and equations defining such an $\mathcal{A}_{\mu\rho}^T$ algorithm, viz.,

$$\begin{aligned}&u(t) = \mu(t) k^0 |x(t)| \\ &\dot{\mu}(t) = -\alpha \operatorname{sgn} \sigma'_w(t) \text{ for } |\mu(t)| \leqslant 1 \\ &\qquad = -w\mu(t) \text{ for } |\mu(t)| > 1, \ |\mu(t_0)| \leqslant 1, \ t \geqslant t_0 \\ &\sigma'_w(t) = \sigma_w(t) + \delta\mu(t) |x(t)| \\ &\sigma_w(t) = -cw(t) - x(t) \\ &T\dot{w}(t) + w(t) = \dot{x}(t)\end{aligned} \tag{7.34}$$

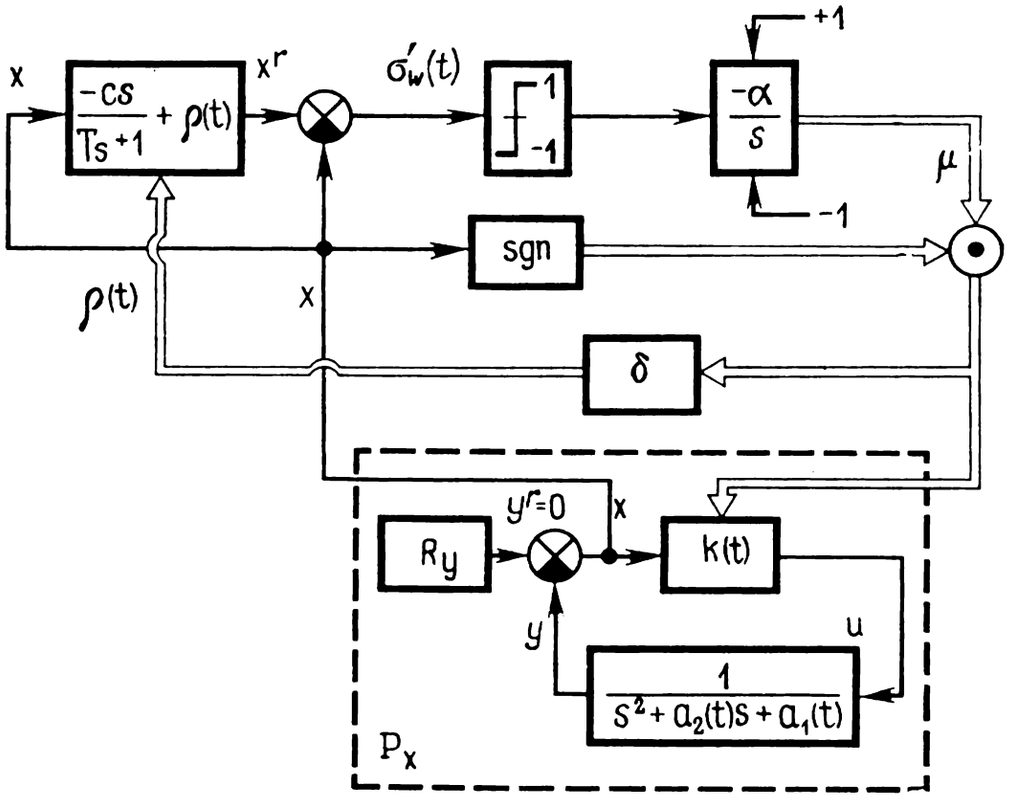

Figure 7.6

As can be seen from these equations and the pertinent block diagrams, this section will be concerned with $\mathcal{A}_{\mu}^{T}(1)$ and $\mathcal{A}_{\mu\rho}^{T}(1)$ control algorithms. We shall evaluate the behavior of the outlined control systems for the $\mathcal{P}_x$-process modeled as

$$\ddot{x}(t)+a_2(t)\dot{x}(t)+a_1(t)x(t)=-u(t) \tag{7.35}$$

where $a_1(t)$ and $a_2(t)$ vary subject to the constraints (4.2). We will aim at determining the conditions under which the $\mathcal{P}_x$-process (7.32), (4.2) is steady controlled.

Examining S_{μ}^{T} Systems

Here we adopt the same scheme of analysis we employed in earlier chapters. As mentioned above, at $T=0$ the S_{μ}^{T} system coincides with the S_{μ} system investigated in Section 6.1. Under the conditions stated in this section a part of controlled processes of the S_{μ} system (x^1 processes) reaches the domain G_0 in a finite time to remain subsequently in the domain G_{δ}. It would be natural to expect that at a sufficiently small values of the time constant T the behavior of controlled processes in the S_{μ}^{T} system will be close to that of S_{μ} system processes. Our analysis here will be aimed at revealing the conditions under which such qualitative coincidence does take place. We shall, of course, have to deal with another domain of the type G_{δ} and other conditions of controlled process convergence to

the origin of the phase space, but it is desirable that all such conditions assume the familiar form at $T = 0$.

The study of the S_μ^T system, generally speaking, is more involved than the study of the S_μ system, since the former configuration is of fourth order rather than third as is the S_μ system. The solutions to the S_μ^T system are generated as the set of functions (x (t), w (t), μ (t)) for $t \geqslant t_0$ and the controlled processes x (t) are the projections of these solutions from $\mathbb{R}^4$ onto $\mathbb{R}^2$. The initial condition stated in (7.33) for μ (t) selects among the feasible solutions the family of V_T solutions on which we shall concentrate below. The S_μ^T system proper is characterized by differential equations with a discontinuous right-hand side [see (7.33)], therefore its solutions will be understood after Filippov [93]. The existence and continuation of these solutions onto the entire semiaxis $t \geqslant t_0$ is also to be established as outlined in this publication.

For any initial conditions on w (t) a detailed analysis is rather complicated even for V_T solutions, therefore we limit ourselves to evaluating a situation with

$$| \dot{x} (t_0) - w (t_0) | \leqslant TB_0 \| x (t_0) \| \tag{7.36}$$

where B_0 is a positive constant. Under some conditions this inequality is valid also for $t > t_0$. It would considerably simplify our further evaluation and the derivation of necessary bounds. Therefore, we wish to formulate and prove this useful fact below.

The proof of (7.36) validity for all t beginning from some finite time instant will be pivoted on the fact that the function $\dot{x}(t)$ is not an arbitrary time function, but rather is defined by a dynamic system whose solutions can be bounded from above by the solutions to the linear dynamic system with bounded coefficients. That is why we are in a position to derive a prior bound such as (7.36) for the difference of $\dot{x}$ (t) and w (t).

Lemma 7.2-1. If for the S_μ^T system (7.16), (7.30)-(7.33), (7.35)

$$1/T - P > 0 \tag{7.37}$$

then for any its V_T solution (x (t) μ (t), w (t)) such that $\| x (t_0) \| \neq 0$ and for an arbitrary number $\varepsilon > 0$ there exists a constant $B_\varepsilon > 0$ and a finite time instant t_ε from which

$$|\dot{x}(t) - w(t)| \leqslant TB_\varepsilon \| x(t) \| \tag{7.38}$$

where B_ε is defined as

$$B_\varepsilon = P_0/(1 - TP) + \varepsilon \tag{7.39}$$

and the constants P_0 and P as

$$P_0 = \max\{\sup_{t\geqslant t_0}[k^0 + |a_1(t)|], \sup_{t\geqslant t_0}|a_2(t)|\} \tag{7.40}$$

$$P = \sup_{t\geqslant t_0} \| A_\mu(t)\| \tag{7.41}$$

the matrix $A_\mu(t)$ being defined in (6.76).

We note that inequality (7.37) relates in a certain manner the variation rate of controlled processes in the S_μ^T system and the variation rate for the derivative bound $w(t)$ obtained in a physical differentiating device.

Proof. Consider a V_T solution $(x(t), \mu(t), w(t))$ of the S_μ^T system that belongs to the family outlined in the conditions of Lemma 7.2-1. Again, as in similar situations above, we verify that if $\| x(t_0) \| \neq 0$, then $\| x(t) \| \neq 0$ for all $t \geqslant t_0$.

For the V_T solution on hand, for all $t \geqslant t_0$, the function

$$\varphi(t) = |\eta(t)|/\| x(t) \| \tag{7.42}$$

is defined and continuous. Here

$$\eta(t) = \dot{x}(t) - w(t) \tag{7.43}$$

Let $\varphi(t_0) = \varphi_0$, then the variation of $\varphi(t)$ for the V_T at hand on the semiaxis $t \geqslant t_0$ is defined as

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} \frac{|\eta(\tau)|^\bullet - \varphi(\tau)\| x(\tau)\|^\bullet}{\| x(\tau)\|} d\tau \tag{7.44}$$

where points as before denote time derivatives of the respective functions:

$$|\eta(\tau)|^\bullet = \dot{\eta}(\tau)\operatorname{sgn}\eta(\tau)$$

$$\| x(\tau)\|^\bullet = \dot{x}(\tau)\operatorname{sgn} x(\tau) + \ddot{x}(\tau)\operatorname{sgn}\dot{x}(\tau)$$

From (7.43) and the equations of S_μ^T system we find that for $t \geqslant t_0$ the function $\eta(t)$ satisfies the equation (the argument t is omitted for convenience)

$$T\dot{\eta} = -\eta - T(k^0\mu |x| + a_1 x + a_2\dot{x}) \tag{7.45}$$

Multiplying both sides of this equation by $\operatorname{sgn}\eta(t)$ yields

$$T|\eta|^\bullet = -|\eta| - T(k^0\mu|x| + a_1 x + a_2\dot{x})\operatorname{sgn}\eta \tag{7.46}$$

Observing here the inequality

$$|k^0\mu|x| + a_1 x + a_2\dot{x}| \leqslant P_0\| x\|$$

valid for every V_T solution and $t \geqslant t_0$, yields the differential inequality

$$T \mid \eta \mid^{\cdot} \leqslant -\mid \eta \mid + TP_0 \parallel x \parallel \tag{7.47}$$

which will be handy in our further considerations.

Let us now obtain a bound of the ratio $\parallel x \parallel^{\cdot}/\parallel x \parallel$. To this end, we observe first of all that

$$\begin{aligned} \mid \parallel x \parallel^{\cdot} \mid &= \mid \mid x \mid^{\cdot} + \mid x \mid^{\cdot} \mid \\ &= \mid \dot{x} \operatorname{sgn} x + \ddot{x} \operatorname{sgn} \dot{x} \mid \leqslant \mid \dot{x} \mid + \mid \ddot{x} \mid = \parallel \dot{x} \parallel \end{aligned}$$

Further we invoke equation (6.76) in the form $\dot{x} = A_\mu x$, which is essentially the vector form of equation (7.35) for the control $u(t)$ given in agreement with (7.16). This vector equation leads to the estimate $\parallel \dot{x} \parallel \leqslant P \parallel x \parallel$ valid for $t \geqslant t_0$, with which we immediately arrive at

$$\mid \parallel x \parallel^{\cdot} \mid / \parallel x \parallel \leqslant P \tag{7.48}$$

This inequality and (7.47) will be used to bound the integrand in equation (7.44). We omit the elementary algebra involved to quote the result

$$\varphi(t) \leqslant \varphi_0 + \int_{t_0}^{t} [P_0 - (1/T - P)\, \varphi(\tau)]\, d\tau, \quad t \geqslant t_0 \tag{7.49}$$

To solve this integral inequality we define the continuously differentiable function $\psi(t)$ as follows:

$$\psi(t) = \varphi_0 + \int_{t_0}^{t} [P_0 - (1/T - P)\, \psi(\tau)]\, d\tau \tag{7.50}$$

and convince ourselves that, given (7.37), $\psi(t) \geqslant \varphi(t)$ for all $t \geqslant t_0$. Subtracting (7.50) from (7.49) yields

$$\varphi(t) - \psi(t) \leqslant -\int_{t_0}^{t} (1/T - P)\, [\varphi(\tau) - \psi(\tau)]\, d\tau$$

In view of (7.37) this inequality immediately gives that $\varphi(t) - \psi(t) \leqslant 0$, hence, for all $t \geqslant t_0$ $\varphi(t) \leqslant \psi(t)$. Recall that $\psi(t)$ for all $t \geqslant t_0$ satisfies the differential equation

$$\dot{\psi} = -(1/T - P)\, \psi + P_0, \quad \psi(t_0) = \varphi(t_0) \tag{7.51}$$

resulted from differentiating (7.50). The solution to this equation is

$$\begin{aligned} \psi(t) = \psi_0 &\exp\{-(1/T - P)\,(t - t_0)\} \\ &+ \int_{t_0}^{t} \exp\{-(1/T - P)\,(t - \tau)\}\, P_0\, d\tau \end{aligned}$$

which after integration takes the form

$$\psi(t) = P_0/(1/T - P) + [\varphi_0 - P_0/(1/T - P)] \times \exp\{-(1/T - P)(t - t_0)\} \quad (7.52)$$

This expression immediately suggests that if $\varphi_0 \leqslant TP_0/(1 - TP)$, then for $t > t_0$ $\varphi(t) \leqslant \psi(t) \leqslant TP_0/(1 - TP)$. Moreover, given an arbitrary small ε, for any initial condition $\varphi_0 > TP_0/(1 - TP)$ there exists a finite time instant $t_\varepsilon \geqslant t_0$ such that $\varphi(t) \leqslant TP_0/(1 - TP) + T\varepsilon$ for all $t \geqslant t_\varepsilon$.

Recalling the expression for $\varphi(t)$ [see (7.42)] we arrive, for $t \geqslant t_\varepsilon$, under the considered conditions at the inequality

$$|\eta(t)| \leqslant T[P_0/(1 - TP) + \varepsilon] \|x(t)\|$$

which coincides with the desired inequality (7.38). Lemma 7.2-1 is proved.

We have proved thereby that beginning from a time t_ε the bound (7.38) holds for every V_T solution of the S_μ^T system provided $1 - TP > 0$.

The said time instant t_ε is individual for each V_T solution. The fact that bound (7.38), heavily involved in our further considerations, is valid for each V_T solution from another, unknown time instant is not of much convenience. Of course, we could start a consideration for a V_T solution from the time t_ε so that all the subsequent results be valid, but this would complicate formulations for many assertions. It is more convenient, without much loss of generality, to confine ourselves to evaluating those V_T solutions for which at the initial time instant (7.36) holds with $B_0 = B_{\varepsilon=0} = P_0/(1 - TP)$. Then for $t > t_0$ by Lemma 7.2-1 we shall have for such V_T solutions

$$|x(t) - w(t)| \leqslant TB_0 \|x(t)\|, \quad |B_0 = P_0/(1 - TP) \quad (7.53)$$

We shall refer to such V_T solutions as V_T^0 solutions and henceforth focus our attention on this class. For these solutions, the following theorems become easier to formulate while the attendant proof becomes versatile.

We continue investigation of S_μ^T systems using the bound (7.53) as a pivotal point. According to the general methodology of binary system construction, the COFB loop is to close the error of this loop. In the case under examination this reduces to ensuring

$$\sigma_w(t) = 0 \quad (7.54)$$

or, which is the same [see (7.32)],

$$w(t) + x(t) = 0 \quad (7.55)$$

The function $w(t)$, however, cannot be expressed through $\dot{x}(t)$ and $\eta(t)$: from (7.30) and (7.43) we have for it $w(t) = \dot{x}(t) - \eta(t)$. Therefore, fulfillment of the equality in (7.55) means also that

$$\sigma(t) = -c\eta(t) \tag{7.56}$$

where as before $\sigma(t) = -[c\dot{x}(t) + x(t)]$.

In agreement with Lemma 7.2-1, given (7.37), we have on the semiaxis $t \geqslant t_0$ for V_T^0 solutions $|\eta(t)| \leqslant TB_0 \|x(t)\|$. Observing this inequality in (7.56) indicates that when the desired equality (7.54) holds the function $\sigma(t)$ satisfies

$$|\sigma(t)| < cTB_0 \|x(t)\| \tag{7.57}$$

This inequality implies that under ideal operation of the COFB loop in the S_μ^T system ensuring $\sigma_w(t) = 0$ from a certain time, the projections of its V_T^0 solutions on the phase space belong to the domain

$$\bar{G}_T = \{x: |\sigma(x)| \leqslant cTB_0 \|x\|\} \tag{7.58}$$

This specifically implies that there is no reckoning on a complete solution to the control problem on hand with $\sigma(t) \equiv 0$ in S_μ^T systems. The best we could hope for is to get an approximate solution by ensuring $x(t) \in \bar{G}_T$. This, however, requires keeping equality (7.54) satisfied, which in general is not an easy task. That this is the case one can easily see from examining the expressions for $\dot{\sigma}_w(t)$ and $\ddot{\sigma}_w(t)$ derived at $\sigma_w(t) = 0$, and both $\sigma_w(t)$ and $\dot{\sigma}_w(t) = 0$, respectively. Straightforward calculations yield

$$\dot{\sigma}_w|_{\sigma_w=0} = -(1/T)[(c+T)\dot{x} + x] \tag{7.59}$$

$$\ddot{\sigma}_w\Big|_{\substack{\sigma_w=0\\ \dot{\sigma}_w=0}} = \frac{1}{T}\left[(c+T)a_1 + \frac{1-(c+T)a_2}{c+T} + k^0 \mu \operatorname{sgn} x\right]x \tag{7.60}$$

The identical zero in $\sigma_w(t) \equiv 0$ would imply that

$$\mu(t) \equiv -\frac{1}{k^0}\left[(c+T)a_1(t) + \frac{1-(c+T)a_2(t)}{c+T}\right]\operatorname{sgn} x(t)$$

which is impossible as $a_1(t)$ and $a_2(t)$ are arbitrary and piecewise continuous, and the parameter α in the COFB law (7.33) is finite.

Under the circumstances it would be only natural to strive at an approximate fulfillment of (7.54) in choosing circuit parameters of an S_μ^T system. Here, as before, we shall understand the said approximation in the sense of meeting

$$|\sigma_w(t)| \leqslant \delta |x(t)| \tag{7.61}$$

at some fixed positive δ. For this inequality being satisfied with the V_T^0 solutions of the S_μ^T system, in view of $\sigma_w = \sigma + c\eta$ we have

$$| \sigma (t) | \leqslant (\delta + cTB_0) \| x (t) \| \tag{7.62}$$

In the phase space to this inequality there corresponds the domain

$$\overline{G}_{\delta_T} = \{x: | \sigma (x) | \leqslant \overline{\delta}_T \| x \|\} \tag{7.63}$$

where

$$\overline{\delta}_T = \delta + cTB_0 \tag{7.64}$$

Thus, approximately meeting the desired inequality means that the controlled processes $x (t) \in \overline{G}_{\delta_T}$

It would be natural to require that such controlled processes converge to the point O_x. In order to evaluate the conditions under which this will be the case we invoke the transformation of Section 6.11. We note that since $\sigma = -(cx + \dot{x})$, then $c | x | - | \dot{x} | \leqslant | \sigma |$ and with $x \in \overline{G}_{\delta_T}$ we have $| \sigma | \leqslant \overline{\delta}_T (| x | + | \dot{x} |)$. These two inequalities yield

$$| \dot{x} | \leqslant | x | (1 + \overline{\delta}_T)/(c - \overline{\delta}_T)$$

provided

$$c - \overline{\delta}_T > 0 \tag{7.65}$$

Under this assumption $\overline{G}_{\delta_T}$ may be embedded into the larger domain G_{δ_T} defined as follows:

$$G_{\delta_T} = \{x: |\sigma (x)| \leqslant \delta_T |x|\} \tag{7.66}$$

where the constant δ_T is related to the aforementioned constants as

$$\delta_T = \overline{\delta}_T (1 + c)/(c - \overline{\delta}_T) \tag{7.67}$$

If we now apply the differential constraint $c\dot{x} + x = -\sigma$ and the inequality $| \sigma | \leqslant \delta_T | x |$ valid as long as $x \in \overline{G}_{\delta_T}$, we shall be able to solve for such controlled processes the problem of convergence to O_x.

Similar to proving Theorem 6.11-1 and Corollary 6.11-1 we establish that the controlled processes converge to O_x when $\delta_T < 1$ or, which is the same, when

$$\overline{\delta}_T < c/(2 + c) \tag{7.68}$$

In addition, the following bound is valid for $t \geqslant t_1$

$$\| x (t) \| \leqslant \frac{c + 1 + \delta_T}{c} \| x (t_1) \| \exp \left\{ - \frac{1 - \delta_T}{c} (t - t_1) \right\} \tag{7.69}$$

where t_1 is the time instant from which $x (t) \in G_{\delta_T}$.

This result will be valid, of course, if $x(t) \in G_{\delta_T}$ for all t starting with a certain time instant. We look at this circumstance in some more detail.

Problem B. In general sense, the domain G_{δ_T} is not invariant in S_μ^T systems. The best we may hope for is conditional invariance or, in accord with Definition 6.7-2, G invariance. The key point therefore is to choose the domain $G \subset G_{\delta_T}$. As such we pick up the domain $\overline{G}_T$ (7.58).

We consider only V_T^0 solutions to the S_μ^T system, therefore the term V solution in Definition 6.7-2 must be replaced with the term V_T^0 solution. The associated conditional invariance of $\overline{G}_T$ for such solutions will be called G_T^0 invariance. The conditions for $\overline{G}_T^0$ invariance of G_{δ_T} are given by the following theorem.

Theorem 7.2-1. Let for positive constants c and δ_T, $0 < \delta_T < 1$, the conditions (7.36)-(7.37) and the relationships

$$k^0 > \overline{k}_T^0 \tag{7.70}$$

$$\alpha \geqslant \frac{2(1 - TB_0)c}{\delta_T - TB_0(1 + \delta_T + c)}(k^0 + \overline{\overline{k}}_T^0) \tag{7.71}$$

($\overline{k}_T^0$, $\overline{\overline{k}}_T^0$ are displayed below) be satisfied for the S_μ^T system (7.16), (7.30), (7.32)-(7.33), (7.35). Then the domain G_{δ_T} is $\overline{G}_T^0$ invariant for such S_μ^T system.

The proof of this theorem could be achieved on the lines traditional for this treatise, therefore we skip over the details and outline only the principal fragments.

We are about to demonstrate in this theorem that for an arbitrary V_T^0 solution $(x(t), \mu(t), w(t))$ to the S_μ^T system, given the initial condition $x(t_0) \in \overline{G}_T$, the controlled process $x(t)$ will not leave the domain G_{δ_T} for $t \geqslant t_0$. To begin with we assume that at t_1 $x(t)$ appears at the boundary ∂G_{δ_T} or G_{δ_T} and prove that then $\mu(t_1) = -\operatorname{sgn} \sigma_w(t_1)$. Then we employ this result to establish that $x(t)$ will not leave G_{δ_T} for $t > t_1$.

In order to prove the first fact we invoke the function

$$\lambda(t) = \sigma(t)/\delta_T \mid x(t) \mid, \quad t \geqslant t_0 \tag{7.72}$$

subject to $\mid x(t_0) \mid \neq 0$.

For each V_T^0 solution, this function can be written as the integral equation

$$\lambda(t) = \lambda(t_0) + \int_{t_0}^{t} \left[\frac{c}{\delta_T}(a_1 - a_2\zeta + \zeta^2) \operatorname{sgn} x + \mu \frac{ck^0}{\delta_T}\right] d\tau \tag{7.73}$$

where $\zeta = (1 + \delta_T\xi)/c$, and $\xi(t) = \lambda(t) \operatorname{sgn} x(t)$.

To simplify our working, assume $x(t_0) \in \partial\overline{G}_T$ and $x(t) \in G_{\delta_T} \setminus G_T$ for $t \in (t_0, t_1]$, where t_1 is the first instant when $x(t_1) \in \partial G_{\delta_T}$.

For $|\lambda(t_0)|$ a straightforward bound results as

$$|\lambda(t_0)| = \frac{|\sigma(t_0)|}{\delta_T |x(t_0)|} = \frac{cTB_0 \| x(t_0)\|}{\delta_T |x(t_0)|} \leqslant \frac{TB_0}{1-TB_0} \frac{1+c}{\delta_T}$$

We note only that $1 - TB_0 > 0$ follows from (7.68) satisfied under the conditions of Theorem 7.2-1.

Assume now that at $t = t_1$, when by agreement $|\lambda(t_1)| = 1$, we have $\mu(t_1) \neq -\operatorname{sgn}\sigma_w(t_1)$. Observing that for t in $(t_0, t_1]$ we have $|\sigma(t)| > cTB_0 \| x(t) \|$ we find that on this time interval $\operatorname{sgn}\sigma(t) = \operatorname{sgn}\sigma_w(t)$ so that under this assumption $t_1 - t_0 < 2/\alpha$.

Denote

$$\bar{\bar{k}}_T^0 = \max_{|\xi| \leqslant 1} \sup_{t \geqslant t_0} |a_1 - a_2\zeta + \zeta^2| \tag{7.74}$$

$$\bar{k}_T^0 = \max_{|\xi| = 1} \sup_{t \geqslant t_0} |a_1 - a_2\zeta + \zeta^2| \tag{7.75}$$

where $\zeta = (1 + \delta_T\xi)/c$.

With this notation and the estimates for $|\lambda(t_0)|$ and $t_1 - t_2$ we obtain from (7.73)

$$|\lambda(t_1)| \leqslant \frac{TB_0}{1-TB_0} \frac{1+c}{\delta_T} + \frac{2c}{\delta_T} (\bar{\bar{k}}_T^0 + k^0) \frac{1}{\alpha}$$

Using (7.71) in this bound leads us to establish that $|\lambda(t_1)| < 1$ contrary to the assumption made, and hence $\mu(t_1) = -\operatorname{sgn}\sigma_w(t_1)$.

To conclude the proof we need to verify that for t close to and following t_1 the equality $\mu(t) = -\operatorname{sgn}\sigma_w(t) = -\operatorname{sgn}\sigma(t) = -\operatorname{sgn}\lambda(t)$ holds true, therefore, substituting $\mu(t) = -\operatorname{sgn}\lambda(t)$ into (7.73) and observing (7.70) we obtain that for these t $|\lambda(t)| < 1$ which implies that $x(t) \in G_{\delta_T}$ and thereby concludes the proof of the theorem.

By way of comment we note that at $T = 0$ (7.70) and (7.71) take on the form usual for S_μ systems with integral COFB.

Theorem 7.2-1 is valid if $\overline{G}_T$ is a domain of attraction for V_T^0 solutions of the S_μ^T system. The following problem is devoted to establishing the sufficient conditions for $\overline{G}_T$ to be attracting.

Problem C. The solution of this problem hinges on the fact that the signs of $\sigma(t)$ and $\sigma_w(t)$ coincide when $x(t) \notin \overline{G}_T$ for some V_T^0 solution to the S_μ^T system at hand. Indeed, $\sigma_w = \sigma + c\eta$, but for a V_T^0 solution $|\eta(t)| \leqslant TB_0 \| x(t) \|$ and since $|\sigma(t)| > cTB_0 \times \| x(t_0) \|$ for $x(t) \notin \overline{G}_T$, then $\operatorname{sgn}\sigma_w(t) = \operatorname{sgn}\sigma(t)$. With this equ-

ality on hand we can put down for $\sigma(t)$ an equation similar to (6.68) (t is dropped throughout)

$$\dot{\sigma} = -\frac{ca_2-1}{c}\sigma - \left(\frac{ca_2-1}{c} - ca_1\right)x - ck^0|x|\operatorname{sgn}\sigma \tag{7.76}$$

which is valid when the controlled process $x(t)$ does not belong to $\bar{G}_T$ on a time interval longer than $2/\alpha$.

From (7.76) we obtain that given (6.67) and (7.70) subject to $\xi = 0$ [this is equivalent to $k^0 > \tilde{k}^0$ with $\tilde{k}^0$ given in (6.94)], either there exists a finite time instant t' when $x(t') \in \partial\bar{G}_T$ or $\lim_{t\to\infty}\sigma(t) = 0$. The last equality is equivalent to $\lim_{t\to\infty}\|x(t)\| = 0$. Hence, $\bar{G}_T$ is a domain of attraction for the S^T_μ system. In passing we have also solved Problem D concerning the asymptotic behavior of those V^0_T solutions for which $x(t) \notin G_{\delta_T}$.

Now we are in a position to state that in an S^T_μ system all controlled processes divide themselves under the aforestated conditions into two families, those of $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. The controlled processes corresponding to $\tilde{V}^0_T$ solutions converge to the origin. Given inequality (7.68) is satisfied, $x^{\mathrm{I}}(t)$ may be exponentially estimated with (7.69) from some time instant.

Thus we have demonstrated that under some conditions a parametrically uncertain $\mathcal{P}_x$-process is $\mathcal{A}^T_\mu$ controlled. The fact that it is steady $\mathcal{A}^T_\mu$ controlled may be established with the argument of Section 6.7. The shorthand expression summarizing this result, provided, of course, the conditions (6.67), (7.36)-(7.37) and (7.68) are satisfied, is as follows:

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}^T_\mu}\left\{\mathcal{F}_c,\ \bar{I}_2;\ \frac{c_m}{\delta_T}\left(\frac{1+\delta_T+c}{c}\right)^2 e^{2P/\alpha}\right\} \tag{7.77}$$

where α is borrowed from Theorem 7.2-1, P from Lemma 7.2-1, and k^0 from (7.70).

To conclude we make another remark on the dependence of x^1 processes on $\mathcal{P}_x$ parameters in the S^T_μ system. No choice of circuit parameters is able to diminish this dependence to an arbitrary level. An indication of this fact is provided in particular by the lower bound of the distance between the points $\pi^t = (c - \bar{\delta}_T\bar{\lambda}\operatorname{sgn}\dot{x},\ 1 - \bar{\delta}_T\bar{\lambda}\times\operatorname{sgn}x)$ and $\pi^0 = (c,\ 1)$ in the parameter space π_x, estimated as

$$\Delta^t_\mu = \|\pi^t - \pi^0\| \geqslant 2cTB_0 \tag{7.78}$$

in the limit as $\delta \to 0$. Here $\bar{\lambda} = \sigma/\bar{\delta}_T\|x\|$ with $|\bar{\lambda}| \leqslant 1$.

S_μ^T System Analysis

Let us look now at the behavior of S_μ^T systems augmented by a static operator feedback (OFB) loop. This implies that we shall examine S_μ^T systems described by (7.34)-(7.35). Again we limit ourselves to analysis of V_T^0 solutions under the assumption of Lemma 7.2-1 being satisfied. The lemma provides for $\eta(t) = \dot{x}(t) - w(t)$ the inequality (7.53) for $t \geqslant t_0$, i.e., $|\eta(t)| \leqslant TB_0 \|x(t)\|$.

It will be noted that the concept of V_T^0 solutions to the S_μ^T system (7.34)-(7.35), the existence of these solutions and their continuity on the entire semiaxis $t \geqslant t_0$ are clarified with the remarks issued for V_T^0 solutions to the S_μ^T system in the foregoing section.

We recall that for $S_{\mu\rho}$ system with complete information (when $T = 0$) we assumed the equality $\sigma'(t) = \sigma(t) + \delta\mu(t)|x(t)| = 0$ is maintained for system's solutions from some instant. This assumption enabled us to determine system's circuit parameters and evaluate the principal properties of system's solutions. In what follows we shall adhere to these lines of reasoning.

Let us for a V_T^0 solution to the $S_{\mu\rho}^T$ system on hand have from some time instant $t_1 \geqslant t_0$

$$\sigma'_w(t) = \sigma_w(t) + \delta\mu(t)|x(t)| = 0 \tag{7.79}$$

Since $|\mu(t)| \leqslant 1$ for V_T^0 solutions we have for $t \geqslant t_1$ $|\sigma_w(t)| \leqslant \delta|x(t)|$. Recalling that $\sigma_w = \sigma + c\eta$ we see that (7.62) is satisfied. In view of the condition (7.65) this is equivalent to $x(t) \in G_{\delta T}$, the domain $G_{\delta T}$ being defined in (7.66). Therefore the asymptotic behavior of controlled processes for such V_T^0 solutions can be examined in much the same way as we did in the previous section. With (7.68) being met, the respective controlled processes converge to O_x and from t_1 their norms obey the bound (7.69).

Equation (7.79) can be employed for establishing the dependence of some controlled processes in the $S_{\mu\rho}^T$ system on $\mathcal{P}_x$ parameters. To demonstrate, let $|x(t)| \neq 0$ for $t \geqslant t_1$; resolve (7.79) for $\mu(t) \times |x(t)|$ with account of $\sigma_w = \sigma + c\eta = -(c\dot{x} + x) + c\eta$, and incorporate the result into the first line of (7.34), then

$$u(t) = -(k^0/\delta)[c\dot{x}(t) + x(t)] - (ck^0/\delta)\eta(t)$$

Represent $\eta(t)$ as

$$\eta(t) = TB_0[\lambda_1(t)\dot{x}(t) + \lambda_2(t)x(t)]$$

with $\lambda_1(t)$ and $\lambda_2(t)$ being some continuous time functions of bounded variation $|\lambda_i(t)| \leqslant 1$, $i = 1, 2$, in view of $|\eta(t)| \leqslant TB_0 \times \|x(t)\|$ for V_T^0 solutions.

Substituting these expressions for $u(t)$ and $\eta(t)$ in (7.35) we establish that for all $t \geqslant t_1$ the controlled processes satisfy the equation

$$\ddot{x}(t) + [a_2(t) + ck^0/\delta + (ck^0TB_0/\delta)\lambda_1(t)]\dot{x}(t) + [a_1(t) + k^0/\delta + (ck^0TB_0/\delta)\lambda_2(t)]x(t) = 0 \tag{7.80}$$

This equation suggests that, as contrasted from $S_{\mu\rho}$ systems, in $S^T_{\mu\rho}$ systems even at constant parameters a_1 and a_2 the differential equation includes variable parameters, the implication being that such system dynamics cannot enjoy asymptotic independence of plant parameters. What we could reckon on in this case is to achieve a slight dependence on parameters ("dissipatism") in the extent estimated from below by letting δ go to zero in equation (7.80). The resultant differential equation, satisfied with the considered controlled processes, has the form

$$c[1 + \lambda_1(t)TB_0]\dot{x}(t) + [1 + \lambda_2 TB_0]x(t) = 0 \tag{7.81}$$

Therefore for the distance $\Delta^t_{\mu\rho}$ in the parameter space π_x from an ideal point $\pi^0 = (c, 1)$ corresponding to the equation $c\dot{x} + x = 0$, to point $\pi^t = (c(1 + \lambda_1 TB^0), 1 + c\lambda_2 TB_0)$ corresponding to (7.81) we have the following sufficient bound:

$$\Delta^t_{\mu\rho} = \| \pi^t - \pi^0 \| \leqslant 2cTB_0 \tag{7.82}$$

Comparing this expression with Δ^t_μ of (7.78) reveals that static OFB can improve the system performance.

The $S^T_{\mu\rho}$ system processes will possess the aforementioned properties if for system's arbitrary V^0_T solution there exists a finite time instant t_1 when $\sigma'_w(t)$ falls off to zero and remains so for subsequent t. If such a finite time instant does not exist for some V^0_T solutions, their asymptotic behavior with t at infinity should be examined. In compliance with our classification, the first problem is to be solved as Problem A, the next as Problem B, and the last as Problem D. We begin our examination with solving Problem B.

Problem B. Assume at $t_1 \geqslant t_0$

$$\sigma'_w(t_1) = \sigma_w(t_1) + \delta\mu(t_1)|x(t_1)| = 0 \tag{7.83}$$

We are to determine the conditions under which (7.79) will be valid for all $t \geqslant t_1$. It is worthwhile to handle this problem as consisting of two cases: (1) $|\mu(t_1)| < 1$ and (2) $|\mu(t_1)| = 1$. We give only the outline of the examination, the details can be restored on the lines of argument used for Theorem 7.1-2.

Case 1. Using (7.34) and (7.43) we can readily write the equation for $\sigma'_w(t)$ variation on the evaluated V_T^0 solution for $t \geqslant t_1$, viz.,

$$\begin{aligned}\dot{\sigma}'_w(t) &= \int_{t_1}^{t} (\dot{\sigma}_w + \delta\dot{\mu}\ \operatorname{sgn} x\dot{x} + \delta\dot{\mu}\ |x|)\, d\tau \\ &= \int_{t_1}^{t} (-c\dot{w} + \dot{x} + \delta\mu\ \operatorname{sgn} x\dot{x} - \alpha\delta\ |x|\ \operatorname{sgn}\sigma'_w)\, d\tau \\ &= \int_{t_1}^{t} (-c\eta/T - \xi_1\dot{x} - \alpha\delta\ |x|\ \operatorname{sgn}\sigma'_w)\, d\tau \qquad (7.84)\end{aligned}$$

where $\xi_1 = 1 - \delta\mu\ \operatorname{sgn} x$. Inspection of the integrand suggests that if with a sufficiently small constant $\Delta > 0$ for t in $(t_1, t_1 + \Delta)$ the function $|\sigma'_w(t)| > 0$, say $\sigma'_w(t) > 0$, and

$$\alpha\delta\ |x(t)| \geqslant -(c/T)\,\eta(t) - \xi_1(t)\,\dot{x}(t) \qquad (7.85)$$

then the signs of the integrand and of $\sigma'_w(t)$ are different (i.e., $\dot{\sigma}'_w(t) \leqslant 0$ on $(t_1, t_1 + \Delta]$) and hence $\sigma'_w(t) = 0$ on $[t_1, t_1 + \Delta]$. By this we have shown that, given (7.83) and (7.85) are satisfied, for the selected V_T^0 solution for $t \geqslant t_1$, and at the same time $|\mu(t)| < 1$, then for such V_T^0 solution we have (7.79), i.e. $\sigma'_w(t) = 0$, for all $t \geqslant t_1$.

Let us rearrange (7.85) in view of the last equality to get $cw(t) = -\xi_1(t)\,x(t)$. Now the derivative $\dot{x} = w + \eta$ can be represented as

$$\dot{x}(t) = -[\xi_1(t)/c]\,x(t) + \eta(t) \qquad (7.86)$$

Substituting this expression in (7.85) yields

$$\alpha\delta\ |x(t)| \geqslant -[c/T + \xi_1(t)]\,\eta(t) + [\xi_1^2(t)/c]\,x(t) \qquad (7.87)$$

Since for the V_T^0 solution at hand $x(t) \in G_{\delta_T}$ and always $|\eta(t)| \leqslant TB_0\,\|x(t)\|$ then for $\eta(t)$ we have

$$|\eta(t)| \leqslant TB_0\,\frac{1+c}{c-\bar{\delta}_T}\,|x(t)|$$

which immediately suggests that (7.87) and, consequently, (7.85) will be valid for $t \geqslant t_1$ provided that

$$\alpha \geqslant \frac{c}{\delta}\ \max_{|\xi|\leqslant 1}\left\{\left(\frac{1+\delta\xi}{c}\right)^2 + \left[1 + T\left(\frac{1+\delta\xi}{c}\right)\right] B_0\,\frac{1+c}{c-\bar{\delta}_T}\right\} \qquad (7.88)$$

where ξ is an arbitrary number with $|\xi| \leqslant 1$. This inequality includes only the parameters of the $S_{\mu\rho}^T$ system, consequently, it can be satisfied by a suitable choice of α.

The fact we have just proved is useful on the condition $|\mu(t)| < 1$ for $t \geqslant t_1$. It is essential therefore to reveal whether this inequality takes place in this case. Assume that on some interval $[t_1, t_2)$,

$t_2 > t_1$, $|\mu(t)| < 1$ and $\sigma'_w(t) = 0$. That such a time interval exists is quite obvious. For t in $[t_1, t_2]$

$$\sigma'_w(t) = \int_{t_1}^{t} \dot{\sigma}'_w(\tau)\,d\tau = \int_{t_1}^{t} (\delta |x| \dot{\mu} - c\dot{w} - \xi_1 \dot{x})\,d\tau = 0$$

whence

$$\int_{t_1}^{t} \delta |x| \dot{\mu}\,d\tau = -\int_{t_1}^{t} (c\dot{w} + \xi_1 \dot{x})\,d\tau \tag{7.89}$$

where $\xi_1 = 1 - \delta\mu \operatorname{sgn} x$.

Because $\sigma'_w(t) = 0$ means $x(t) \in G_{\delta_T}$, for such motion $x(t)$ experiences no sign reversal and, consequently, the integrand on the right-hand side of (7.89) is continuous.

Assume now that at $t = t_2$ the function $\mu(t)$ attains its boundary value, for definiteness let $\mu(t_2) = 1$; the case of -1 may be examined in a similar manner. Then it is quite obvious that there exists a small number $\Delta > 0$ such that

$$\int_{t_2-\Delta}^{t_2} (c\dot{w} + \xi_1 \dot{x})\,d\tau < 0 \tag{7.90}$$

In view of integrand's continuity we get with the mean value theorem

$$c\dot{w}(\theta) + \xi_1(\theta)\,\dot{x}(\theta) < 0 \tag{7.91}$$

where θ is a point from $[t_2 - \Delta, t_2]$. Letting Δ go to zero leads us to

$$c\dot{w}(t_2) + [1 - \delta \operatorname{sgn} x(t_2)]\,\dot{x}(t_2) \leqslant 0 \tag{7.92}$$

This inequality implies that for t close to t_2 with $t > t_2$ the continuously differentiable function $v(t) = cw(t) + [1 - \delta \operatorname{sgn} x(t)] \times x(t)$ obeys

$$v(t) \leqslant 0 \tag{7.93}$$

ince by assumption $v(t_2) = 0$.

If we now assume that for such $t > t_2$ $\sigma'_w(t) < 0$, then $\mu(t) = 1$ so that in view of (7.34)

$$\sigma'_w(t) = \int_{t_2}^{t} \dot{\sigma}'_w\,d\tau = \int_{t_2}^{t} -[c\dot{w} - (1 - \delta \operatorname{sgn} x)\,\dot{x}]\,d\tau$$

$$= -\int_{t_2}^{t} \dot{v}(\tau)\,d\tau = -v(t)$$

The last is true in view of $v(t_2) = 0$. Recalling that above we have proved that $v(t) \leqslant 0$ we obtain for the considered t that $\sigma'_w(t) \geqslant 0$ which contradicts the above assumption.

If we now assume that $\sigma'_w(t) > 0$ for $t > t_2$, then in view of (7.34) $\mu(t) < 1$ for t close to t_2, which brings our examination to the initial situation.

It remains to consider the case of $\sigma'_w(t) = 0$ and $\mu(t) = 1$ for t close to t_2 with $t > t_2$. Under these conditions $\dot{v}(t_2) = 0$ and since $\dot{\sigma}'_w(t_2) = -\dot{v}(t_2)$, then $\dot{\sigma}'_w(t_2) = 0$. Therefore the direction in which $\sigma'_w(t)$ varies can be inferred in this case from analysis of its second derivative at $t = t_2$. To obtain this derivative we first derive some useful relationships. From $\sigma'_w(t_2) = 0$ we get

$$-cw(t_2) = \xi x(t_2), \quad \xi = 1 - \delta \operatorname{sgn} x(t_2) \tag{7.94}$$

From $\dot{\sigma}'_w(t_2) = 0$ we get $-c\dot{w}(t_2) = \xi \dot{x}(t_2)$ and, in view of $\dot{w} = \eta/T$,

$$-c\eta(t_2)/T = \xi \dot{x}(t_2) \tag{7.95}$$

Next we use $\dot{x} = \eta + w$ to establish with (7.94)-(7.95) that

$$\eta(t_2) = \frac{T\xi}{c(c+T\xi)} x(t_2) \tag{7.96}$$

$$\dot{x}(t_2) = -\frac{\xi}{c+T\xi} x(t_2) \tag{7.97}$$

under the obvious assumption of $c + T\xi > 0$.

Now we see that $\ddot{\sigma}'_w(t_2)$ has sense and reads as

$$\ddot{\sigma}'_w(t_2) = -[c\dot{w}(t_2) + \xi \ddot{x}(t_2)] \tag{7.98}$$

Since $T\ddot{w} = \dot{\eta}$ and in view of (7.45) at $\mu = 1$ $\dot{\eta} = -\eta/T - (k^0 \mid x \mid + a_1 x + a_2 \dot{x})$ and $\ddot{x} = -a_2 \dot{x} - a_1 x - k^0 \mid x \mid$, we substitute these expressions into (7.98) with account of (7.96) and (7.97) and after some algebra arrive at the desired expression (t_2 is suppressed throughout)

$$\ddot{\sigma}'_w = \frac{c+T\xi}{T} \left\{ k^0 \mid x \mid + \left[\left(\frac{\xi}{c+T\xi} \right)^2 - \left(\frac{\xi}{c+T\xi} \right) a_2 + a_1 \right] x \right\} \tag{7.99}$$

From this expression it is not hard to infer that $\ddot{\sigma}'_w(t_2) > 0$ provided $\mid x(t_2) \mid > 0$ and

$$k^0 > \hat{k}^0_T = \max_{\xi = 1 \pm \delta} \sup_{t \geqslant t_0} \left| a_1 - a_2 \left(\frac{\xi}{c+T\xi} \right) + \left(\frac{\xi}{c+T\xi} \right)^2 \right| \tag{7.100}$$

Note in passing that if at some t_1 $x(t_1) = 0$ for $x(t_1) \in G_{\delta_T}$, then the V^0_T solution for subsequent $t > t_1$ has the form $(O_x, \mu(t), 0)$ and, consequently, $\sigma'_w(t) = 0$.

The inequality $\ddot{\sigma}'_w(t_2) > 0$ just proved enables us to state that for t close to t_2, $t > t_2$, the function $\sigma'_w(t)$ increases under the investigated condition, that is for the said t the inequality $\dot{\sigma}'_w(t) > 0$ must hold.

Turning to the differential equation for $\mu(t)$ we find that for $t > t_2$ the derivative $\dot{\mu}(t) = -\alpha$, i.e., $\mu(t)$ decreases, consequently, $\mu(t) < 1$ for $t > t_2$. Thus, we again return to the initial situation with $|\mu(t)| < 1$ and therefore $\sigma'_w(t) = 0$. Recycling the above considerations we establish that given (7.88) and (7.100) the equality $\sigma'_w(t) = 0$ holds for all subsequent t following t_1 when it occurs. In the circumstances, for such a V_T^0 solution $|\mu(t)| \leqslant 1$ for all $t \geqslant t_1$, the set of instants with $|\mu(t)| = 1$ is at least enumerable.

Case 2. Here we are faced with a number of possibilities. The one to be taken up first is characterized by $\sigma'_w(t_1) = 0$, $\mu(t_1) = 1$ and $\sigma'_w(t) > 0$ for $t > t_1$. Accordingly, in view of $\dot{\mu} = -\alpha \operatorname{sgn} \sigma'_w$ the function $\mu(t)$ diminishes, i.e., $\mu(t) < 1$ for $t > t_1$. Invoking (7.84) it is an easy matter to see that, given (7.88), $\sigma'_w(t) = 0$ for t close to t_1 and following this instant. This reduces the examination of this situation to *Case 1* above.

A similar situation takes place under the following conditions:

$$\sigma'_w(t_1) = 0, \quad \mu(t_1) = -1, \quad \text{and} \quad \sigma'_w(t) < 0 \quad \text{for} \quad t > t_1$$

A third situation is described by $\sigma'_w(t_1) = 0$, $\mu(t_1) = 1$, and $\sigma'_w(t) < 0$ for t close to t_1 with $t > t_1$. This version needs thorough examination. Although somewhat in advance, we note that analysis of the next situation with $\sigma'_w(t_1) = 0$, $\mu(t_1) = -1$ and $\sigma'_w(t) > 0$ for $t > t_1$ parallels the analysis of the situation at hand, therefore we confine ourselves to examining this case only. First of all we note that this situation would be impossible if $\sigma'_w(t) > 0$ for $t < t_1$ as then $\mu(t_1) \neq 1$. Consequently, for t close to but smaller than t_1 there should be $\sigma'_w(t) \leqslant 0$. Under the conditions stated $\sigma'_w(t) = -cw(t) - [1 - \delta \operatorname{sgn} x(t)]\, x(t)$ and in a small neighborhood of t_1 this function is continuous. Hence at $t = t_1$ it reaches a maximum $\sigma'_w(t_1) = 0$, which in turn means that $\dot{\sigma}'_w(t_1) = 0$. Hence its behavior for $t > t_1$ can be inferred from $\ddot{\sigma}'_w(t_1)$ for which we have (7.99). Given (7.100) this derivative is positive (we assume, of course, that $|x(t_1)| > 0$) and therefore for t posterior and close to t_1 $\sigma'_w(t) > 0$. This, however, contradicts the initial assumption of $\sigma'_w(t) < 0$, $t > t_1$. Hence under the stated conditions the situation on hand is unfeasible. Summarizing all the above developments and cases we see that given (7.88) and (7.100) the equality $\sigma'_w(t) = 0$ holds at all t beginning with the instant it occurs for the first time. Introducing the set

$$\Gamma_w = \{(x, \mu, w): \sigma_w + \delta\mu |x| = 0, \ |\mu| \leqslant 1\} \tag{7.101}$$

we can summarize the above considerations as the following statement.

Theorem 7.2-2. Given (7.68), (7.88), (7.100) and the conditions of Lemma 7.2-1, then the set Γ_w is invariant for V_T^0 solutions of the $S_{\mu\rho}^T$ system (7.34), (7.35).

We have demonstrated that if at $t = t_1$ some V_T^0 solution $[x(t), \mu(t), w(t)]$ of the $S_{\mu\rho}^T$ system belongs to Γ_w, the respective controlled process $x(t) \in G_{\delta_T}$ for all $t \geqslant t_1$ and hence if (7.68) is satisfied the trajectory converges to the origin O_x subject to the bound (7.69). Theorem 7.2-2 therefore is central to investigation of $S_{\mu\rho}^T$ systems, but to make it useful we need to establish that the set Γ_w is a domain of attraction for V_T^0 solutions to the $S_{\mu\rho}^T$ system (7.34)-(7.35). A definition of the respective conditions makes up the content of Problem C that follows.

Problem C. To begin with, note that the set Γ_w will be a domain of attraction for $S_{\mu\rho}^T$ system's V_T^0 solutions if at $t = t_0$ for any $V_T^0 \notin \Gamma_w$ there is a finite time when $V_T^0 \in \Gamma_w$ or else $\lim\limits_{t\to\infty} V_T^0 \in \Gamma_w$.

We examine the problem at hand in two steps. The first step will be devoted to establishing that G_{δ_T} in (7.66) is a domain of attraction. Next we find conditions under which a V_T^0 solution attains Γ_w, i.e. where $\sigma_w'(t)$ vanishes.

To prove attraction we shall use the coincidence of signs of $\sigma_w'(t)$ and $\sigma(t)$ for $x(t) \notin G_{\delta_T}$. This follows at once from $\sigma_w' = \sigma + c\eta + \delta\mu \mid x \mid$, the expression for G_{δ_T}, and inequality (7.53) valid for V_T^0 solutions.

Assume $V_T^0 \notin \Gamma_w$ for all $t \geqslant t_0$, then $\sigma_w'(t)$ experiences no sign reversals on this trajectory. Let for definiteness $\sigma_w'(t) < 0$, $t \geqslant t_0$, and the respective $x(t) \notin G_{\delta_T}$. Above we have noted that $\operatorname{sgn} \sigma_w'(t) = \operatorname{sgn} \sigma(t)$ and if on the interval $[t_0, t_0 + 2/\alpha]$ the controlled process does not attain the domain G_{δ_T}, then for $t \geqslant t_0 + 2/\alpha$ the $\sigma(t)$ satisfies (7.76). Analysis of this equation as before suggests that given $\inf\limits_{t\geqslant t_0} ca_2(t) \geqslant 1$ and $k^0 > k_T^0$, where k_T^0 is defined in (7.75) at $\xi = 0$, for the controlled process on hand $x(t)$, there exists a finite t_1 such that either $x(t_1) \in \partial G_{\delta_T}$, or $\lim\limits_{t\to\infty} \sigma(t) = 0$.

The last equality implies, as has been demonstrated already, that in the circumstances $\lim\limits_{t\to\infty} \| x(t) \| = 0$. Consequently the domain G_{δ_T} is attracting subject to (6.67) and $k^0 > k_T^0$. It is essential that under the assumptions made $\mu(t) = -\operatorname{sgn} \sigma_w'(t) = 1$ for $t \geqslant t_1$.

Now we go over to the second step of the proof. Consider for $t \geqslant t_1$ on the V_T^0 solution the continuous function

$$\lambda(t) = \lambda(x(t)) = \sigma(x(t))/\delta_T \mid x(t) \mid \qquad (7.102)$$

subject to $\lambda(t_1) = -1$. It is not hard to verify that for this function the following integral equation can be written

$$\lambda(t) = -1 + (c/\delta_T) \int_{t_1}^{t} [(a_1 - a_2 \zeta + \zeta^2) \operatorname{sgn} x + k^0] \, d\tau \qquad (7.103)$$

where $\xi(t) = \lambda(t) \operatorname{sgn} x(t)$ and $\zeta = (1 + \delta_T \xi)/c$. Notice that this equation coincides with (7.73) on letting $\mu = 1$ there.

Analysis of this equation readily suggests that for $k^0 > \bar{\bar{k}}_T^0$ $\lambda(t)$, $t \geqslant t_1$, monotonously increases at a finite rate. Consequently, there is a finite time t_2 when $\lambda(t_2) = 1$. By virtue of (7.102) this implies that $\sigma(t) = \lambda(t) \delta_T \mid x(t) \mid > \delta_T \mid x(t) \mid$ for $t > t_2$. In other words the controlled process $\boldsymbol{x}(t)$ leaves G_{δ_T} for $t > t_2$. Outside G_{δ_T} the signs of $\sigma'_w(t)$ and $\sigma(t)$ coincide, that is for $t > t_2$ $\sigma'_w(t) > 0$ which contradicts our initial assumption of $\sigma'_w(t) < 0$ for all $t \geqslant t_0$. Therefore there has to exist a time instant t' when $\sigma'_w(t') = 0$, i.e. when the V_T^0 solution attains the set Γ_w. Q.E.D. The foregoing development leads us to formulate the following theorem.

Theorem 7.2-3. Let for the $S_{\mu\rho}^T$ system (7.34)-(7.35) at some constants $c > 0$ and $\delta > 0$ the following conditions be satisfied: (6.67), (7.36), (7.37), (7.68), (7.88), and

$$k^0 > \max(\bar{\bar{k}}_T^0, k_T^0, \hat{k}_T^0) \qquad (7.104)$$

where $\bar{\bar{k}}_T^0$ is given by (7.74), $\hat{k}_T^0$ by (7.100) and k_T^0 by (7.75) at $\xi = 0$. Then the $\mathcal{P}_x$-process is $\mathcal{A}_{\mu\rho}^T$ controlled and all the controlled processes in such $S_{\mu\rho}^T$ system may be divided into two families, that of $\boldsymbol{x}^{\mathrm{I}}(t)$ and $\boldsymbol{x}^{\mathrm{II}}(t)$ processes. For $\boldsymbol{x}^{\mathrm{I}}$ processes, the bound (7.69) holds beginning from a certain time instant, whereas $\boldsymbol{x}^{\mathrm{II}}(t) \notin G_{\delta_T}$.

Under the conditions of Theorem 7.2-3 every controlled process in the $S_{\mu\rho}^T$ system examined converges to the origin O_x. The proof of the fact that point O_x is partially stable, i.e. the $\mathcal{P}_x$-process is steady $\mathcal{A}_{\mu\rho}^T$ controlled, may be effected on the lines of the argument used in Section 6.7. As a result it can be verified that given the conditions (6.67), (7.36), (7.37), and (7.68), the following shorthand expression holds:

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_{\mu\rho}^T} \left\{ \mathcal{F}_c, \bar{I}_2; \frac{c_m}{\delta_T} \left(\frac{1 + \delta_T + c}{c} \right)^2 e^{2P/\alpha} \right\} \qquad (7.105)$$

where the notation is the same as before, α is taken from (7.88), and k^0 from (7.104).

To close the section a few remarks are in order. (*i*) Some increase in the constant B_0 to a value of B_ε retains the validity of the results established above for V_T solutions of the $S_{\mu\rho}^T$ system (7.34)-(7.35) as well. (*ii*) In contrast to linear control systems, binary systems

operating under imperfect information on the controlled process are capable of high performance control with bounded gains in the primary feedback control loop and with continuous manipulated variable. All the transients involved vary monotonously and for a portion of them the dependence on controlled process parameters arbitrarily varying over bounded ranges can be diminished to a certain level which is lower the less erroneous the data acquisition means are. (*iii*) Changing from $\mathscr{A}_{\mu}^{T}$ (1) and $\mathscr{A}_{\mu\rho}^{T}$ (1) algorithms to $\mathscr{A}_{\mu}^{T}$ (2) and $\mathscr{A}_{\mu\rho}^{T}$(2) enables one to omit the constraining assumption (6.67), i.e. expand the class of $\mathscr{P}_x$-processes controlled in this way, preserving the qualitative nature of the above findings.

Similar conclusions can be drawn for other types of imperfections in data acquisition or with due account of the approximation errors entailed in mathematical model build up for the controlled process or the entire system, and with the use of other control laws in COFB loop.

7.3 $S_{\mu\rho}$ (1) Systems with Inertial Coordinate-Operator Feedback

This section will examine binary control systems od Section 6.8 as augmented by static operator feedback (OFB) define in (7.13). We establish the key effects and features in S_{μ} and S_{μ}^{ρ} systems performance associated with OFB, and derive the necessary design relationships for such systems. Figure 7.7 presents the block diagram for $S_{\mu\rho}$ (1) systems to be evaluated below. The respective $\mathscr{A}_{\mu\rho}$ control algorithm is defined by

$$
\begin{aligned}
u(t) &= \mu(t)\, k^{0} \mid x(t) \mid \\
\dot{\mu}(t) &= -\alpha\,[\mu(t) + \operatorname{sgn}\sigma'(t)] \qquad (7.106)\\
\sigma'(t) &= \sigma(t) + \delta\mu(t) \mid x(t) \mid \\
\sigma(t) &= -[c\dot{x}(t) + x(t)], \quad t \geqslant t_0
\end{aligned}
$$

To obtain the equations of motion for the $S_{\mu\rho}$ system on hand, equations (7.106) must be completed with equation (7.35).

Motivation in evaluating such systems and goals aimed at by introducing operator feedback have been discussed in depth in Sections 6.8 and 7.1. In the qualitative study of the $S_{\mu\rho}$ system (7.35), (7.106) we shall confine ourselves to a schematic outline of principal results, the respective details can be recovered with the developments of Sections 6.8 and 7.1.

Like in similar developments above, this analysis of the $S_{\mu\rho}$ system (7.35), (7.106) is based on revealing the conditions which ensure the equality $\sigma'(t) = \sigma(t) + \delta\mu(t) \mid x(t) \mid = 0$ on V solut-

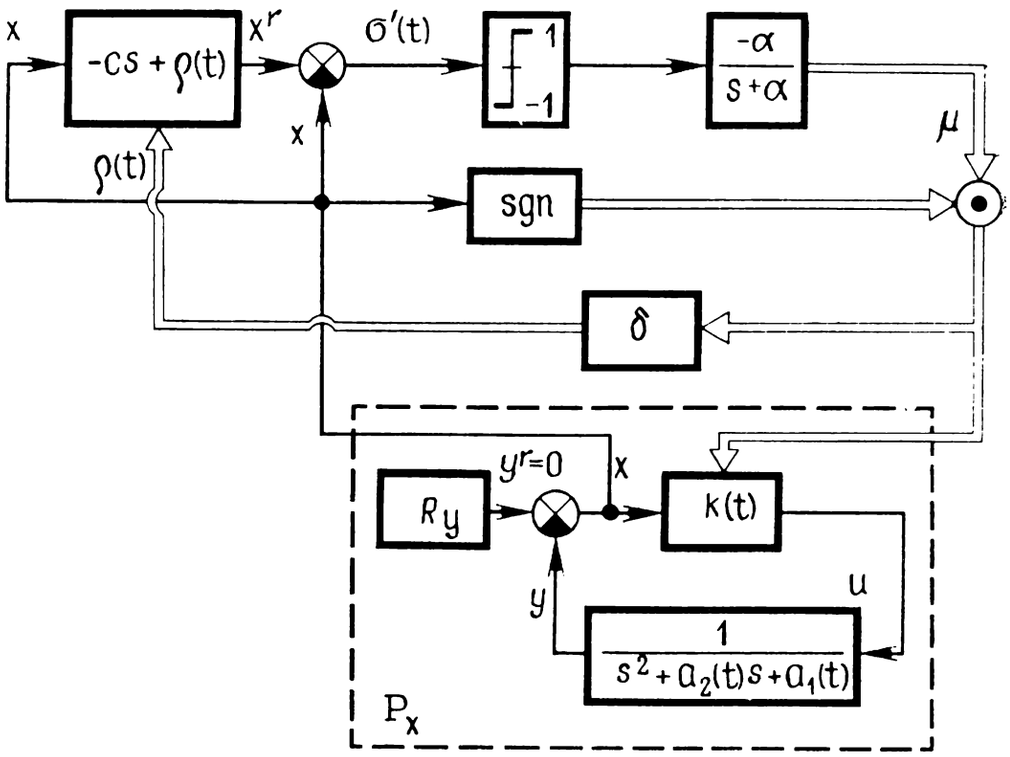

Figure 7.7

ions $[x\ (t),\ \mu\ (t)]$ beginning from some time instant. Since for V solutions we always have $|\ \mu\ (t)\ | \leqslant 1$, the desired equality $\sigma'\ (t) = 0$ implies that the controlled process $x\ (t) \in G_\delta$, where $G_\delta = \{x: |\ \sigma\ (x)\ | \leqslant \delta\ |\ x\ |\}$. Therefore the problem of convergence of such processes to the origin O_x is decided by Theorem 6.7-1 and Corollary 6.7-1 subject to the bound (6.58).

It will be noted that if any controlled process $x\ (t) \notin G_\delta$ then $\sigma'\ (t)$ and $\sigma\ (t)$ have identical signs and whether or not these controlled processes converge to G_δ is decided on the basis of Theorems 6.8-2 and 6.8-3. Recall that by these theorems for one part of controlled processes (more specifically for x^{II} processes) $\lim\limits_{t\to\infty} \|x\ (t)\| = 0$, and for the other part there exists a finite t_1 when $\sigma'\ (t_1) = 0$. Therefore the central topic of the investigation concerns conditions to maintain this equality for all posterior t. Solving this problem in affirmative implies that the set (7.18), viz., $\Gamma = \{(x,\ \mu)$: $\sigma\ (x) + \mu\delta\ |\ x\ | = 0,\ |\ \mu\ | \leqslant 1\}$ is invariant on V solutions of the $S_{\mu\rho}$ system (7.35), (7.106). Such invariance conditions are outlined in the theorem that follows.

Theorem 7.3-1. If for some positive number h, $0 < h < 1$, and positive constants c and δ, $0 < \delta < 1$

$$(1 - h)\ k^0 = \bar{k}^0 \tag{7.107}$$

$$\alpha \geqslant (k^0 + \bar{\bar{k}}^0)\ c/\delta \tag{7.108}$$

where the constants $\bar{k}^0$ and $\bar{\bar{k}}^0$ are defined by (6.86) and (6.72) respectively, then the set Γ is invariant on the V solutions of the system (7.35) and (7.106).

Some key points of the proof follow. First of all we verify that $|x(t)| > 0$ for $t \geqslant t_0$. Then the variation of $\sigma'(t)$ for $t \geqslant t_0$ (notice that $\sigma'(t_0) = 0$) is defined by

$$\sigma'(t) = \int_{t_0}^{t} [c(a_1 - a_2 \xi_\lambda/c + \xi_\lambda \xi_\mu/c^2)\, x + \mu c k^0 |x| - \alpha\delta |x| (\mu + \operatorname{sgn} \sigma')]\, d\tau \qquad (7.109)$$

where $\xi_\lambda(t) = 1 + \delta\lambda(t) \operatorname{sgn} x(t)$, $\xi_\mu(t) = 1 - \delta\mu(t) \operatorname{sgn} x(t)$, and where $\lambda(t) = \sigma(t)/\delta |x(t)|$.

Making use of the fact that the sum of two last terms in the integrand of (7.109) is linear in μ we see that given (7.107) and (7.108) the integrand in (7.109) is always opposite in sign to $\sigma'(t)$. Therefore, if $\sigma'(t_0) = 0$, then under the conditions of Theorem 7.2-3 for all $t \geqslant t_0$ we have $\sigma'(t) = 0$. This equality implies that the set Γ is invariant on V solutions of the $S_{\mu\rho}$ system (7.35), (7.106).

Comparison of (7.107) and (7.108) with the relationships (6.85) and (6.87) of Theorem 6.8-1 for an S_μ system with inertial COFB suggests that the incorporation of operator feedback means considerable suppression of gains in the CFB and COFB loops and avoids tradeoff in their choice.

It will be noted also that since h can be taken arbitrarily small the conditions of Theorem 7.2-3 are equivalent to that of Theorem 7.1-2 for $S_{\mu\rho}$ systems with integral COFB. Therefore for the $S_{\mu\rho}$ system at hand we can formulate a direct analog of Theorem 7.1-3 to define conditions of $\mathcal{P}_x$-process $\mathcal{A}_{\mu\rho}$ controllability. Since all remarks made in Section 7.1 concerning the $S_{\mu\rho}$ system with integral COFB are valid here as well, we can also prove that the $\mathcal{A}_{\mu\rho}$ control thus outlined is steady in nature. Summarizing the results just stated we may formulate them in shorthand as follows:

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_{\mu\rho}} \left\{ \mathcal{F}_c,\ I_2;\ \frac{c_m}{\delta} \left(\frac{1+c+\delta}{c} \right)^2 e^{2P/\alpha} \right\} \qquad (7.110)$$

where the notation is as before while k^0 and α are borrowed from Theorem 7.3-1.

To close this section it is worth noting that since the equality $\sigma'(t) = 0$, fulfilled for the S_μ system in question from some time instant, is equivalent to the equality (7.24) for the $S_{\mu\rho}$ system with integral COFB, all the properties of operator feedback established in Section 7.1 (Problem F) for the latter system are valid for the former. Specifically, the control law (7.106) can secure a parametrical

static error, whereas asymptotic independence of parameters does not take place, consequently the control problem is not solved completely. A qualitative insight into the behavior of controlled processes in the $S_{\mu\rho}$ system (7.35), (7.106) for some fixed $\mathcal{P}_x$ parameters within (4.2) is given in Fig. 7.4.

7.4 Quasicontinuous Control Algorithm

In this section we perform a qualitative examination of $S_{\mu\rho}$ system behavior with a new type of $\mathcal{A}_{\mu\rho}$ control algorithms. Application of these binary control algorithms to the class of $S_{\mu\rho}$ control systems is motivated by an offer of a set of properties for the controlled processes which, generally speaking, cannot be warranted over the class of S_{μ} systems.

The principal aim of employing such control algorithms is as follows. In the $S_{\mu\rho}$ systems evaluated in the foregoing sections the variation of operator variable $\mu(t)$ was described by a differential equation with a discontinuous right-hand side. An integral COFB law had three surfaces of discontinuity [$\sigma'(x, \mu) = 0$, $|\mu| = 1$], whereas for the inertial COFB law there was only one such surface, viz., $\sigma'(x, \mu) = 0$. Under certain conditions, such $S_{\mu\rho}$ systems experience sliding modes on these discontinuity surfaces. Although these sliding modes occur outside the feedback control loop closed via the plant, and the control function $u(t)$ is a continuous function of time, it may be desirable for a number of situations to somehow avoid these sliding modes. We outline two reasons which may cause it to be a requirement.

In numerical modeling of $S_{\mu\rho}$ systems with discontinuous COFB laws or in implementing the respective $\mathcal{A}_{\mu\rho}$ algorithms with logic elements the designer is faced with a certain hardships caused by the lack of well developed and efficient methods for assigning difference models with required properties to the differential dynamic systems with discontinuous right-hand side. It is desirable therefore to evaluate $\mathcal{A}_{\mu\rho}$ algorithms that prevent sliding modes from occurring in the closed-loop control system, or if such occur then they should have a finite duration on the main phase of transients. In this case, motions in this control system are modeled by differential equations with continuous or piecewise continuous right-hand side and the respective difference equations are put in correspondence to them by traditional techniques. We call such control algorithms quasicontinuous algorithms and dwell on this topic below.

The block diagram of the quasicontinuous $S_{\mu\rho}$ system to be examined is given in Fig. 7.8. The respective equations of the quasicon-

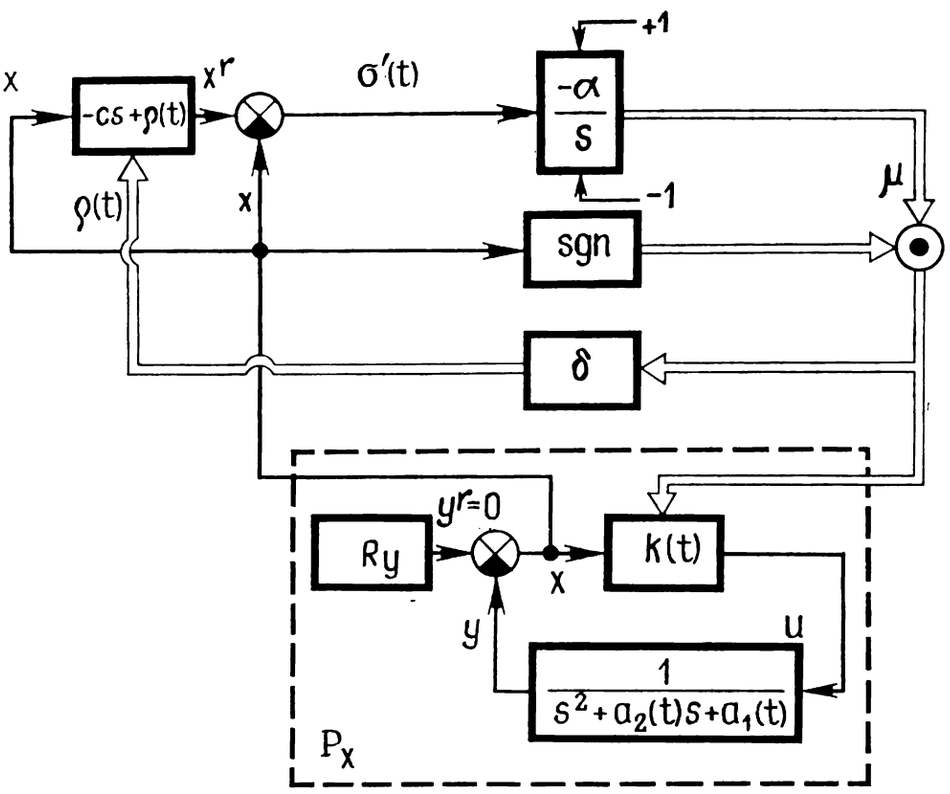

Figure 7.8

tinuous $\mathscr{A}_{\mu\rho}$ algorithm [105, 108, 109] have the form

$$u(t) = \mu(t)\, k^0 \mid x(t) \mid$$

$$\mu(t) = -\alpha\sigma'(t) \quad \text{for} \quad \mid \mu(t) \mid \leqslant 1$$

$$= -w\mu(t) \quad \text{for} \quad \mid \mu(t) \mid > 1, \quad \mid \mu(t_0) \mid \leqslant 1, \quad t \geqslant t_0 \qquad (7.111)$$

$$\sigma'(t) = \sigma(t) + \delta\mu(t) \mid x(t) \mid$$

$$\sigma(t) = -(c\dot{x}(t) + x(t))$$

Supplementing these equations with the one for the $\mathscr{P}_x$-process

$$\ddot{x}(t) + a_2(t)\,\dot{x}(t) + a_1(t)\,x(t) = -u(t), \quad t \geqslant t_0 \qquad (7.112)$$

having the parameters $a_1(t)$ and $a_2(t)$ varying within (4.2) we obtain the equations of motion for the $S_{\mu\rho}$ system on hand. The system, as these equations suggest, refers to $S_{\mu\rho}$ (1) systems. One of equations in (7.111) has a discontinuous (at $\mid \mu \mid = 1$) right-hand side. To render such a control algorithm quasicontinuous we need to reveal conditions ensuring that every solution of the $S_{\mu\rho}$ system (7.111)-(7.112) either misses the boundary of discontinuity or, if it occurs there, leaves the boundary in a finite time. In what follows we shall be concerned with the search for such conditions.

In this analysis of $S_{\mu\rho}$ systems we again content ourselves with examining two types of solutions, namely, V solutions for which $\mid \mu(t_0) \mid \leqslant 1$, and V' solutions for which $\mid \mu(t_0) \mid < 1$. This examination differs from other similar treatments in that for V solut-

ions the equality $\sigma'(t) = 0$, in general, no longer holds. This fact can be verified by analyzing the expression for $\dot{\sigma}'(t)$ which on the assumption of $|\mu(t)| < 1$ and $|x(t)| > 0$ has the form

$$\dot{\sigma}' = c\left(a_1 - a_2\frac{\xi_\lambda}{c} + \frac{\xi_\lambda\xi_\mu}{c^2}\right)x + \mu ck^0 |x| - \alpha\delta |x| \sigma' \qquad (7.113)$$

where $\xi_\lambda(t) = 1 + \delta\lambda \operatorname{sgn} x$, $\xi_\mu = 1 - \delta\mu \operatorname{sgn} x$, and $\lambda = \sigma/\delta |x|$.

For $\sigma'(t) = 0$ this equation takes the form

$$\delta' = c\left[a_1 - a_2(\xi_\mu/c) + (\xi_\mu/c)^2\right]x + \mu ck^0 |x| \qquad (7.114)$$

which obviously can be of any sign for finite α. Therefore in step with the general methodology of binary system analysis developed in this book we shall aim at fulfilling $|\sigma'(t)| \leqslant \bar{\delta} |x(t)|$ at some constant $\bar{\delta} > 0$. For definiteness we shall further assume $\bar{\sigma} = \delta/2$, so that this inequality becomes

$$|\sigma'(t)| \leqslant |x\ t)|\ \delta/2 \qquad (7.115)$$

This inequality suggests that the scheme of argument used above to advantage may be employed for this analysis as well. The development thus will be as follows. First, for every V solution $[x(t), \mu(t)]$ such that at $t = t_1 \geqslant t_0$ (7.115) holds we need to derive relationships with which this inequality is satisfied at subsequent t (Problem B). Then we need to define conditions which ensure the existence of a finite time instant t_1 when (7.115) is satisfied (Problem C). Once Problems B and C have been solved, we should investigate the asymptotic behavior of those controlled processes for which (7.115) is valid and those for which it breaks down from some time instant, i.e. $|\sigma'(t)| > |x(t)|\ \delta/2$ (Problems A and D). Finally we are to establish the behavior of the controlled processes in a small neighborhood of the origin O_x (it will be distinct from the other control systems) and examine the dependence of the processes on $\mathcal{P}_x$ parameters (Problems E and F). In addition to the aforelisted problems, we are to establish the quasicontinuous nature of the $\mathcal{A}_{\mu\rho}$ algorithm (7.111) — that is the subject matter of Problem G.

Problem A. This problem is the simplest of the foregoing list and will be tackled first. Assume the inequality (7.115), i.e. $|\sigma'(t)| = |\sigma(t) + \delta\mu(t)|\ x(t)|\ | \leqslant |x(t)|\ \delta/2$ holds for some V solution of the $S_{\mu\rho}$ system (7.111)-(7.112) on an interval $[t_1, t_2]$, $t_2 > t_1$. Since $|\mu(t)| \leqslant 1$ for $t \in [t_1, t_2]$ we have in this interval

$$\sigma(t) \leqslant |x(t)|\ 3/2 \qquad (7.116)$$

This inequality may be interpreted as $x(t) \in G_{3\delta/2}$ where the domain $G_{3\delta/2}$ is defined as

$$G_{3\delta/2} = \{x: |\sigma(x)| \leqslant |x|\ 3\delta/2 \qquad (7.117)$$

Whenever $x(t) \in G_{3\delta/2}$ in $[t_1, t_2]$ we may invoke Theorem 6.7-1. It is an easy matter to verify the following result derived from this theorem.

Corollary 7.4-1. Suppose for some constant $c > 0$ and $\delta > 0$, on some V solution $[x(t), \mu(t)]$ for $t \in [t_1, t_2]$, $t_2 > t_1 \geqslant t_0$, we have $x(t) \in G_{3\delta/2}$, then for $t \in [t_1, t_2]$

$$\| x(t) \| \leqslant \frac{1+c+3\delta/2}{c} \| x(t_1) \| \exp\left[-\frac{1-3\delta/2}{c} (t-t_1) \right] \tag{7.118}$$

If in addition $3\delta/2 < 1$, then for all $t \in [t_1, t_2]$

$$\| x(t) \| \leqslant \frac{1+c+3\delta/2}{c} \| x(t_1) \| \tag{7.119}$$

This corollary enables us to establish the convergence of controlled processes to the origin O_x if $x(t) \in G_{3\delta/2}$ for all $t \geqslant t_1$.

Problem C. Recall that this problem is to determine conditions ensuring that there exists a finite time t_1 when inequality (7.115) is satisfied for any V solution. More accurately the respective conditions are defined in the theorem that follows. To formulate it we need to introduce the definitions of open and closed spheres centered on the origin O_x, specifically, for any constant $r > 0$ $U_r = \{x: \| x \| < r\}$ defines an open sphere, and $\bar{U}_r = \{x: \| x \| \leqslant r\}$ a closed sphere.

Theorem 7.4-1. Let $G_0 = \{x: \sigma(x) = 0\}$ be a domain of $\mathcal{O}$-attraction for the VS system (6.67), then for any $r > 0$ and every V solution of the $S_{\mu\rho}$ system (7.111)-(7.112) there exists a finite time t_r such that either $x(t_r) \in G_{3\delta/2}$ or $x(t_r) \in \bar{U}_r$.

Saying this another way, the theorem states that, for example, when the condition (6.67) is satisfied and the domain G_0 is $\mathcal{O}$-attracting for the VS system (6.65), every controlled process in the $S_{\mu\rho}$ system at hand either appears in a neighborhood of O_x, however small, and thus solves Problem D, or attains the domain $G_{3\delta/2}$. Notice that the theorem contains no mention of the process behavior for $t > t_r$.

The connection between the $S_{\mu\rho}$ system (7.111)-(7.112) and the VS system (6.65) is worth elucidating. Suppose for this purpose that for $t \geqslant t_0$ $x \notin \bar{U}_r$ for some constant $r > 0$ and

$$| \sigma'(x, \mu) | > | x | \delta/2 \tag{7.120}$$

for some μ with $| \mu | \leqslant 1$. Define also a number $\bar{\delta} > 0$ so that

$$| \sigma'(x, \mu) | \leqslant \delta \| x \| \leqslant | x | \delta/2 \tag{7.121}$$

As in similar situation before, convince ourselves that for $| \mu | \leqslant 1$

$$c | \dot{x} | - (1+\delta) | x | \leqslant | \sigma'(x, \mu) | \leqslant \bar{\delta}(| x | + | \dot{x} |)$$

whence for $c-\bar{\delta}>0$ we obtain a bound for $|\dot{x}|$, viz.,

$$|\dot{x}| \leqslant |x| (1+\delta+\bar{\delta})/(c-\bar{\delta})$$

Using this bound we convert the inequality $|\sigma'| \leqslant \bar{\delta} \|x\|$ into $|\sigma'| \leqslant \bar{\delta} |x| (1+\delta+c)/(c-\bar{\delta})$. Therefore, defining the constant $\bar{\delta}$ as

$$\bar{\delta} = \frac{\delta}{2} \frac{c}{1+c+3\delta/2} \tag{7.122}$$

satisfies the required inequalities (7.121). Using these inequalities it is not hard to see that if $|\sigma'| > |x| \delta/2$ then

$$|\sigma'| > \bar{\delta} \|x\| \tag{7.123}$$

This bound enables us to establish that if for $t \geqslant t_0$ $x \in U_r$ and (7.120) is satisfied then for $t \geqslant t_0$

$$|\sigma'(t)| > \bar{\delta} r \tag{7.124}$$

But then in view of (7.111) for $t \geqslant t_1$ where

$$t_1 = t_0 + 2/\alpha\bar{\delta} r \tag{7.125}$$

we obtain

$$\mu(t) = -\operatorname{sgn} \sigma'(t) \tag{7.126}$$

The inequality (7.124) indicates that $\sigma'(t)$ retains its sign for considered t and since $\sigma'(t) = \sigma(t) + \delta\mu(t) |x(t)|$ and for $t \geqslant t_1$ $\sigma'(t) = \sigma(t) - \delta |x(t)| \operatorname{sgn} \sigma'(t)$, for $t \geqslant t_1$ we have $\operatorname{sgn} \sigma'(t) = \operatorname{sgn} \sigma(t)$. Substituting this expression into the equation of motion of the $S_{\mu\rho}$ system (7.111)-(7.112) we find that the controlled process in question satisfies the VS system equation (6.65). Now we turn to proving Theorem 7.4-1.

Proof. Consider for $t \geqslant t_0$ a V solution $[x(t), \mu(t)]$ to the $S_{\mu\rho}$ system (7.111)-(7.112). Suppose that for this V solution the statement of the theorem is invalid, i.e. $\|x(t)\| > r$ and $|\sigma(t)| > |x(t)| 3\delta/2$ for all $t \geqslant t_0$. The second inequality leads to $|\sigma'(t)| > |x(t)| \delta/2$. But above we have proved that under these conditions for all $t \geqslant t_1 = t_0 + 2/\alpha\bar{\delta} r$ with $\bar{\delta}$ defined in (7.122) the controlled process $x(t)$ is determined by the VS system equation (6.65). By the conditions of the theorem such controlled processes either converge to the origin, i.e. $\lim_{t\to\infty} \|x(t)\| = 0$, or there exists a finite time t_2 such that $\sigma(t_2) = 0$. The last equality means that $\sigma'(t_2) = 0$ which contradicts (7.124) satisfied under the assumption made. Consequently, there is a time t_r when $x(t_r) \in \overline{U}_r$ or $|\sigma'(t_r)| \leqslant |x(t_r)| \delta/2$. Q.E.D.

Now with the proven theorem on hand and with Theorem 6.7-4 on $\mathcal{O}$-attraction for the VS system (6.65) we are able to formulate a corollary with an exact formulation of Problem C solution.

Corollary 7.4-2. Let all the conditions of Theorems 6.7-4 and 7.4-1 be satisfied and r be a positive number, then for every V solution $[x(t), \mu(t)]$ of the $S_{\mu\rho}$ system (7.111)-(7.112) there exists a finite time t_r such that $\| x(t_r) \| \leqslant r$ or

$$| \sigma [x(t_r)] + \delta\mu(t_r) | x(t_r) | | \leqslant | x(t_r) | \delta/2$$

Problem B. This is devoted to evaluating the conditions for the domain $G_{3\delta/2}$ to be conditionally invariant. Denote the set

$$\Gamma_{1/2} = \{(x, \mu): | \sigma(x) + \delta\mu | x | | \leqslant | x | \delta/2, \ | \mu | \leqslant 1\} \tag{7.127}$$

The construction of this set appears natural with account of Theorem 7.4-1. Indeed, this theorem states the conditions for a V solution of the $S_{\mu\rho}$ system (7.111)-(7.112) to appear in the set $\Gamma_{1/2}$. If in subsequent motion $V \in \Gamma_{1/2}$, then we may employ Corollary 7.4-1 for analyzing the asymptotic behavior of the respective controlled processes. Problem B is solved by way of the following theorem.

Theorem 7.4-2. Let r be a positive number, and with some constants $c > 0$ and $\delta > 0$ for the $S_{\mu\rho}$ system (7.111)-(7.112) we have

$$k^0 > \sup_{t \geqslant t_0} | a_1 - a_2\zeta + \zeta^2 | \tag{7.128}$$

for all ξ with $| \xi | = 3/2$, and

$$\alpha \geqslant \frac{2(1 + c + 3\delta/2)}{r\delta^2} (k^0 + \sup_{t \geqslant t_0} | a_1 - a_2\zeta + \zeta^2 |) \tag{7.129}$$

for all ξ with $| \xi | \leqslant 3/2$, where $\zeta = (1 + \delta\xi)/c$. Then every V solution belonging to $\Gamma_{1/2}$ at $t = t_1$ belongs to this set at all $t \in [t_1, t_2]$, $t_2 > t_1 \geqslant t_0$ if $x(t) \notin \overline{U}_r$ for $t \in [t_1, t_2]$.

The choice of parameter α is seen to depend on $r > 0$ and its value varies inversely with r. This fact essentially separates quasicontinuous algorithms from other binary control algorithms. This theorem along with Corollary 7.4-1 and Theorem 7.4-1 warrants that in a finite time every control process enters the sphere $_r$. Subsequent behavior of such process $x(t)$ calls for a separate study. What perhaps is clear is that r predetermines accuracy of control problem solution which is better the higher the value of α. This subject will be taken up at length while solving Problem G, thus far we are going to prove (7.128) and (7.129).

Proof. Let $[x(t), \mu(t)]$ be some admissible V solution of the $S_{\mu\rho}$ system (7.111)-(7.112), i.e. $\| x(t) \| > r$ for $t \in [t_1, t_2]$ and $| \sigma'(t_1) | \leqslant | x(t_1) | \delta/2$. We are to demonstrate that $| \sigma'(t) | \leqslant | x(t) | \delta/2$ for all $t \in [t_1, t_2]$. Assume the contrary is true, namely, there exists

a time t' and a positive number Δ such that $(t', t' + \Delta) \subset [t_1, t_2]$ and

$$| \sigma' (t') | = | x (t') | \delta/2 \tag{7.130}$$

and for $t \in (t', t' + \Delta)$

$$| \sigma' (t) | > | x (t) | \delta/2 \tag{7.131}$$

Let Δ be sufficiently small so that $x (t) \neq 0$ for $t \in (t', t' + \Delta)$. Such a Δ must exist as otherwise $x (t') = 0$ and hence $\| x (t) \| = 0$ contrary to the condition of the theorem. With such a Δ we may write the function

$$\lambda' (t) = \sigma' (t)/\delta | x (t) |, \quad t \in [t', t' + \Delta] \tag{7.132}$$

with $\lim\limits_{t \to t'} | \lambda' (t) | = 1/2$ in view of (7.130). Using this function we rearrange the readily verifiable inequality $| \sigma | \leqslant | \sigma' | + \delta | x |$ to the form valid for $t \in [t', t' + \Delta]$, namely

$$| \sigma (t) | \leqslant [1 + | \lambda' (t) |] \delta | x (t) | \tag{7.133}$$

It is straightforward to see that for the modulus of $\sigma = -c\dot{x} - x$ we may always write the bound

$$| \sigma | \geqslant c | \dot{x} | - | x | = c \| x \| - (1 + c) | x | \tag{7.134}$$

From inequalities (7.133) and (7.134) we can derive the lower bound of $| x (t) |$ in $[t', t' + \Delta]$, viz.,

$$| x (t) | \geqslant \frac{c}{1 + c + [1 + | \lambda' (t) |] \delta} \| x (t) \|$$

which on account of $\| x (t) \| > r$ becomes

$$| x (t) | > \frac{cr}{1 + c + [1 + | \lambda' (t) |] \delta}, \quad t \in [t', t' + \Delta] \tag{7.135}$$

Finally, substituting (7.135) into (7.130) and (7.131) leads to the bound, again for $t \in [t', t' + \Delta]$,

$$| \sigma' (t) | > \frac{\delta}{2} \frac{cr}{1 + c + [1 + | \lambda' (t) |] \delta} \tag{7.136}$$

This bound indicates in particular that $\sigma' (t)$ retains its sign on the considered time interval. Thus far we have not examined the magnitude of $\mu (t)$ for $t \in [t', t' + \Delta]$, now we turn to this aspect, dividing it for the sake of consideration into Case 1 of $| \mu (t') | < 1$ and Cases 2 and 3 of $| \mu (t') | = 1$. We investigate them one by one, assuming that $\sigma' (t) > 0$ for $t \in [t', t' + \Delta]$.

Case 1. If $|\mu(t')| < 1$ then for a sufficiently small Δ the inequality $|\mu(t)| < 1$ will be true for $t \in [t', t' + \Delta]$, therefore for such t in view of (7.111) we shall have $\dot{\mu}(t) = -\alpha\sigma'(t)$. As usually introduce the function $\lambda(t) = \sigma(t)/\delta |x(t)|$ which is obviously defined on the considered time interval and related to $\lambda'(t)$ of (7.132) by

$$\lambda(t) = \lambda'(t) - \mu(t) \tag{7.137}$$

From this equation and the function $\lambda'(t)$ itself it is straightforward to conclude that $\lim\limits_{t \to t'} |\lambda(t)| \leqslant 3/2$.

Let us now evaluate the variation of $\lambda'(t)$ in (7.132) for the considered V solution for $t \geqslant t'$. The equation of motion for the $S_{\mu\rho}$ system, equation (7.137), and the fact of $\dot{\mu}(t) = -\alpha\sigma'(t)$ under the above assumption, taken together, lead to the following equation for $t \geqslant t'$:

$$\begin{aligned}
\lambda'(t) &= \lambda'(t') + \int_{t'}^{t} \frac{\dot{\sigma}' - \lambda'\delta |x|^{\cdot}}{\delta |x|} d\tau \\
&= \lambda'(t') + \int_{t'}^{t} \frac{\dot{\sigma} + \delta\dot{\mu} |x| - \delta\lambda' |x|^{\cdot} + \delta\mu |x|^{\cdot}}{\delta |x|} d\tau \\
&= \lambda'(t') + \int_{t'}^{t} \left(\frac{\dot{\sigma} - \delta\lambda |x|^{\cdot}}{\delta |x|} - \alpha\sigma' \right) d\tau
\end{aligned}$$

The first term in the integrand was computed before; incorporating the result we go over to the equation

$$\begin{aligned}
\lambda'(t) = \lambda'(t') + \int_{t'}^{t} \Big\{ \frac{c}{\delta} \Big[a_1 - a_2 \frac{\xi_\lambda}{c} + \Big(\frac{\xi_\lambda}{c} \Big)^2 \Big] \operatorname{sgn} x \\
+ \mu k^0 c/\delta - \alpha\sigma' \Big\} d\tau
\end{aligned} \tag{7.138}$$

where $\xi_\lambda(t) = 1 + \delta\lambda(t) \operatorname{sgn} x(t)$.

Note now that given (7.136) and (7.129) the integrand is opposite in sign to $\lambda'(t)$ for $t \geqslant t'$. Therefore, for t close and posterior to t' the function $|\lambda'(t)|$ is not increasing. Hence for $t > t'$ $|\lambda'(t)| \leqslant 1/2$. This inequality contradicts the assumption (7.131) of $|\lambda'(t)| > 1/2$, therefore in Case 1 for $t > t'$ there must be $|\sigma'(t)| \leqslant |x(t)| \times \delta/2$.

Case 2. Suppose now that $\mu(t') = \operatorname{sgn} \sigma'(t')$. This case is rather easy and reduces to Case 1. Indeed, since for $t \geqslant t'$ (7.136) holds true, then by virtue of (7.111) $|\mu(t)| < 1$ for t sufficiently close to and following t'. Therefore, equation (7.138) holds along with the conclusions drawn in the analysis of this equation.

Case 3. Let $\mu(t') = -\operatorname{sgn} \sigma'(t')$, then by virtue of (7.136) and $S_{\mu\rho}$ system's equations of motion the function $\mu(t) = -\operatorname{sgn} \sigma'(t)$

for all $t \in [t', t' + \Delta]$. Now for the shown t equation (7.138) for $\lambda'(t)$ may be replaced with

$$\lambda'(t) = \lambda'(t') + \frac{c}{\delta} \int_{t'}^{t} \left\{ \left[a_1 - a_2 \frac{\xi_\lambda}{c} + \left(\frac{\xi_\lambda}{c} \right)^2 \right] \operatorname{sgn} x - k^0 \operatorname{sgn} \lambda' \right\} d\tau \tag{7.139}$$

where $\operatorname{sgn} \lambda' = \operatorname{sgn} \sigma'$.

Given (7.128) is satisfied and a sufficiently small $\Delta > 0$, the sign of the integrand in (7.139) is opposite to that of $\lambda'(t)$, consequently $|\lambda'(t)|$ diminishes for the said t. This conclusion contradicts the initial assumption (7.131). This proves Theorem 7.4-2 and solves Problem B.

Theorems 7.4-1, 7.4-2 and Corollaries 7.4-1, 7.4-2 indicate the conditions that ensure for every controlled process in the $S_{\mu\rho}$ system (7.111)-(7.112) attainment of the sphere $\overline{U}_r$ in a finite time. We now wish to demonstrate that under some condition such controlled processes remain at all subsequent t in some other sphere $\overline{U}_R \supset \overline{U}_r$. This assertion will be proved in Problem E.

Problem E. Consider for $t \geqslant t_r$ a V solution $[x(t), \mu(t)]$ to the $S_{\mu\rho}$ system on hand subject to the initial condition $x(t_r) \in \overline{U}_r$. Since for $x(t) \in \overline{U}_r$ Theorem 7.4-2 is invalid, the exact behavior of $x(t)$ for $t > t_r$ is difficult to evaluate. Therefore we content ourselves with sufficient and, generally speaking, rough estimates. The reasoning employed below follows along similar lines to those used for Lemma 6.7-2.

Denote $R > r$ as

$$R = \frac{2c_m}{\delta} \left(\frac{1 + c + 3\delta/2}{c} \right)^2 r \exp(2P/\alpha\bar{\delta}r) \tag{7.140}$$

where $c_m = \max(1, c)$, δ is defined in (7.122), constant P is computed by (7.41), and α is borrowed from Theorem 7.4-2.

We wish to demonstrate that given Theorems 7.4-1, 7.4-2 and Corollaries 7.4-1, 7.4-2, the controlled process concerned $x(t) \in \overline{U}_R$ for all $t \geqslant t_r$. We are going to provide this statement assuming the converse, that is, under the stated conditions there exists a finite time $t_R > t_r$ such that $\| x(t_R) \| = R$ and $\| x(t) \| > R$ for $t > t_R$. Then there is a time t_1 with $t_r \leqslant t_1 < t_R$ such that $\| x(t_1) \| = r$ and $x(t)$ belongs to the open spherical layer $U_R \setminus \overline{U}_r$ for all $t \in (t_1, t_R)$, where as usual the slant line means "less than".

Consider the interval $[t_1, t_2]$ where

$$t_2 - t_1 = 2/\alpha\bar{\delta}r \tag{7.141}$$

and from (7.122) $\bar{\delta} = \delta c/2\,(1 + c + 3\delta/2)$.

Let us verify that in this time interval $\| x(t) \| < R$. To demonstrate, for the controlled process on hand we can write

$$\dot{x}(t) = A_\mu(t)\, x(t), \quad t \geqslant t_1$$

so that for all $t \in [t_1, t_2]$

$$\| x(t) \| \leqslant \| x(t_1) \| \exp[P(t_2 - t_1)] = r \exp(2P/\alpha\bar{\delta}r) \tag{7.142}$$

where P is given in (7.41).

It will be seen that the number of the right-hand side of (7.142) is less than R introduced by (7.140), therefore $x(t_2) \in U_R \setminus \overline{U}_r$ and hence $t_2 < T_R$. Now if at a point t on $[t_1, t_2]$ $| \sigma'(t') | \leqslant | x(t') | \delta/2$, then under the conditions of Theorem 7.4-2 for all $t > t'$ we shall have $x(t) \in G_{3\delta/2}$. This allows us to employ the bound (7.119), viz., $\| x(t) \| \leqslant \| x(t') \| (1 + c + 3\delta/2)/c$, $t > t'$. Applying (7.142) to the right-hand side of this inequality at $t = t'$ we obtain for all $t \geqslant t_1$

$$\| x(t) \| \leqslant \frac{1 + c + 3\delta/2}{c}\, r \exp(2P/\alpha\bar{\delta}r) \tag{7.143}$$

Again we see that the right-hand side of (7.143) is less than R in (7.140), hence in this case $\| x(t) \| \in U_R \setminus \overline{U}_r$.

Suppose now that for all $t \in [t_1, t_2]$ $| \sigma'(t) | > | x(t) | \delta/2$. But then in view of $\| x(t) \| > r$ we may write inequality (7.124), i.e. $| \sigma'(t) | > \bar{\delta}r$ for $t \in [t_1, t_2]$, where $\bar{\delta}$ is given in (7.122). Accordingly, from (7.111) we find that at $t = t_2$ there must be $\mu(t_2) = -\operatorname{sgn} \sigma'(t_2)$. Consequently, for $t \geqslant t_2$ the controlled process on hand is defined by the VS system equation (6.65). Under the conditions of Corollary 7.4-2 such a controlled process either in a finite time has to attain the sphere $\overline{U}_r$ or there exists a time t_3 such that $| \sigma'(t_3) | \leqslant | x(t_3) | \delta/2$. It is important to note that in the first case for all $t \in [t_2, t_3)$ we have

$$\| x(t) \| \leqslant \frac{2(1 + c + 3\delta/2)}{\delta}\, \frac{c_m}{c} \| x(t_2) \| \tag{7.144}$$

where $c_m = \max(1, c)$.

The validity of this bound follows from the fact that for the VS system under the conditions of Theorem 6.7-4 always $| \sigma(t_2) | \geqslant | \sigma(t) |$. On the other hand, always $| \sigma(x) | \leqslant c | \dot{x} | + | x | \leqslant c_m \| x \|$, in addition $\sigma'(t) = \sigma(t) - \delta | x(t) | \operatorname{sgn} \sigma'(t)$ on the time interval $[t_2, t_3)$, hence for the said t $| \sigma(t) | > | x(t) | 3\delta/2$.

Applying the differential constraint $\dot{x} = -(\sigma + x)/c$ and the bounds derived above in succession yields

$$\| x(t) \| = | \dot{x}(t) | + | x(t) | \leqslant [| \sigma(t) | + | x(t) |]/c$$
$$+ | x(t) | = [| \sigma(t) | + (1+c) | x(t) |]/c < | \sigma(t) |$$
$$\times [1 + (1+c)\, 2/3\delta]/c \leqslant \frac{3\delta + 2(1+c)}{\delta c} c_m \| x(t_2) \|$$

The last inequality is seen to coincide with (7.144). To obtain a bound for $\| x(t) \|$ we make use of (7.142) and (7.144). As a result for all $t > t_2$ we get

$$\| x(t) \| \leqslant \frac{2(1+c+3\delta/2)}{\delta} \frac{c_m}{c} r \exp(2P/\alpha\bar{\delta}r) \tag{7.145}$$

Again, comparing the right-hand side of this expression with R from (7.140) we convince ourselves that either $\| x(t) \| \in U_R$ when $x(t)$ reaches the sphere $\overline{U}_r$ or $\| x(t_3) \| \in U_R \setminus \overline{U}_r$ when $| \sigma'(t_3) | \leqslant | x(t_3) |\, \delta/2$, therefore t_3 belongs to $[t_2, t_R)$.

Finally, from the time instant t_3, under the conditions of Theorem 7.4-2 and Corollary 7.4-1 we have for the controlled process the bound $\| x(t) \| \leqslant \| x(t_3) \| (1 + c + 3\delta/2)/c$ until the process reaches the sphere $\overline{U}_r$. Incorporating the bound for $\| x(t_3) \|$ from (7.145) into this inequality yields for $t > t_1$

$$\| x(t) \| \frac{2c_m}{\delta} \left(\frac{1+c+3\delta/2}{c} \right)^2 r \exp(2P/\alpha\bar{\delta}r) = R \tag{7.146}$$

contrary to the assumption of $\| x(t) \| > R$ for $t > t_R$ with some finite t_R. Thus we have proved the following assertion solving Problem E.

Theorem 7.4-3. Let r be a positive number and for the $S_{\mu\rho}$ system (7.111)-(7.112) at some constants $c > 0$ and $\delta > 0$ there hold the condition (6.67), relationship (6.60) at $\xi = 0$, inequality (7.119), and relations (7.128)-(7.129). Then for every V solution $[x(t), \mu(t)]$ to this system, $x(t_r) \in \overline{U}_r$ means $x(t) \in \overline{U}_R$ for all $t \geqslant t_r$, where R is defined in (7.140).

It should be noted that the bound (7.146) can be improved by removing the factor $2/\delta$ and letting $\bar{\delta} = (\delta/2)\, c/(2 + c)$ to get

$$R = c_m \left(\frac{1+\delta+c}{c} \right)^2 r \exp[2P(1+c)/\alpha r(1-\delta)c] \tag{7.147}$$

Theorem 7.4-1 produces essentially a bound such as (4.4b), and since at a sufficiently large parameter α the number r may be arbitrarily small, in accord with the earlier introduced terminology the $\mathcal{P}_x$-process can be said to be steady $\mathcal{A}_{\mu\rho}$ quasi-controlled with

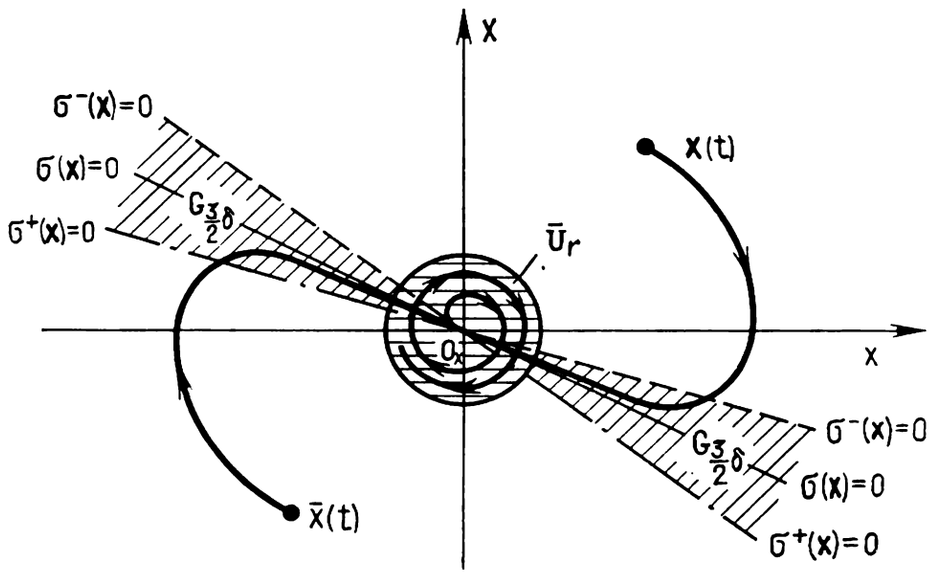

Figure 7.9

the control algorithm (7.111) subject to the condition (6.67). Thus, given (6.67) the situation may be defined in shorthand as

$$\mathcal{P}_x \in \mathcal{QC}_{\mathcal{A}_{\mu\rho}(1)}\{\mathcal{F}_c,\ I_2;\ N\} \tag{7.148}$$

where for every α and the attendant r the constant

$$N = c_m \left(\frac{1+\delta+c}{c}\right)^2 \exp \frac{2P(1+c)}{\alpha r (1-\delta)\, c}$$

This, of course, implies that at every finite α the considered system is dissipative only, i.e. the control problem is solved R-accurately with R as given in (7.147).

A qualitative insight into the behavior of certain controlled processes in such a control system with $\mathcal{P}_x$ parameters a_1 and a_2 fixed within (4.2) is given in Fig. 7.9.

Now we wish to cast a brief glance at how the controlled processes of the $S_{\mu\rho}$ system (7.111)-(7.112) depend upon $\mathcal{P}_x$ parameters. It is quite obvious that this topic is meaningful to consider only subject to $x\,(t) \in G_{3\delta/2}$. This property, however can be warranted only on a short time interval until the controlled process enters the sphere $\overline{U}_r$. On such a time interval, all the remarks made earlier in similar situation for $x \in G_\delta$ are valid, i.e. on this interval the said dependence diminishes with δ. The parameter α, however, increases, and in the limit as $\delta \to 0$ $\alpha \to \infty$, that is, at $\delta \to 0$ the $S_{\mu\rho}$ system (7.111)-(7.112) degenerates into the VS system (6.67).

For a sufficiently small r such that given $\| x \| \leqslant R\ (r)$ the requirements of the accuracy of control are satisfied, we may say, though conditionally to a certain degree, that the $S_{\mu\rho}$ system in question possesses properties close to those of $S_{\mu\rho}$ systems with other COFB laws, but is advantageous over them in that for $|\mu\,(t)| < 1$ none

of the control loop exhibits sliding modes. The last fact follows from the continuous right-hand sides of the equations of motion in this case. Hence, what remains to be clarified is conditions ensuring $|\mu(t)| < 1$ which would imply quasicontinuity. This is Problem G.

Problem G. Taken formally, solving this problem means that for every V' solution to the $S_{\mu\rho}$ system (7.111)-(7.112) the variable $\mu(t)$ is at its boundary value $|\mu| = 1$ at most in one time interval. Notice that $\mu(t)$ appearing on the boundary $|\mu| = 1$ means a sliding mode to occur. For an accurate formulation of the respective conditions we resort to a term introduced by the definition that follows. Recall that S_c denotes a closed-loop control system.

Definition 7.4-1. We call a time interval $[t_3, t_4]$ a μ-constant interval in $[t_1, t_2]$ for a V solution $[x(t), \mu(t)]$ of an S_c system if $[t_3, t_4] \subset [t_1, t_2]$, and $\mu(t) \equiv -1$ or $\mu(t) \equiv 1$ for $t \in [t_3', t_4']$ and there is no interval $[t_3', t_4']$ such that $[t_3, t_4] \subset [t_3', t_4'] \subset [t_1, t_2]$, $[t_3, t_4] \neq [t_3', t_4']$ and $\mu(t) \equiv -1$ or $\mu(t) \equiv 1$ for $t \in [t_3', t_4']$.

With this definition, solution of Problem G reduces to establishing that for V' solutions of $S_{\mu\rho}$ system (7.111)-(7.112) a μ-constant time interval is unique. The respective sufficient conditions are given in the following statement.

Theorem 7.4-4. If with positive constants c, r, and δ the relationships (7.128)-(7.129) are met for the $S_{\mu\rho}$ system (7.111)-(7.112) and

$$k^0 > 2\bar{k}^0 \tag{7.149}$$

with $\bar{k}$ from (6.86), then for every V' solution $[x(t), \mu(t)]$ to this system there is at most one μ-constant time interval in $[t_0, t_r)$ if $x(t) \notin \bar{U}_r$ for all $t \in [t_0, t_r]$.

The direct proof of this theorem is fairly unwieldy, so we take instead a course with three lemmas preceding the major discussion. The first of them, Lemma 7.4-1, defines the sign of $\sigma'(t)$ when $\mu(t)$ enters the boundary.

Lemma 7.4-1. If for some V solution $[x(t), \mu(t)]$ to the $S_{\mu\rho}$ system (7.111)-(7.112), $|\mu(t)| < 1$ for t close to t' with $t_0 \leqslant t < t'$ and $|\mu(t')| = 1$, then for this V solution

$$\mu(t')\,\sigma'(t') \leqslant 0 \tag{7.150}$$

Proof. Consider a V solution satisfying the lemma conditions and assume, without loss of generality, that $\mu(t') = 1$. Now suppose the converse, i.e. that $\mu(t')\,\sigma'(t') > 0$ instead of (7.150). Then by the continuity of $\sigma'(t)$ for t close to t' and $t < t'$ $\sigma'(t) > 0$. Since in the circumstances $\dot{\mu} = -2\sigma'$ the derivative $\dot{\mu}$ is negative that is for the said t $\mu(t') < 1$ which contradicts the lemma. Q.E.D.

The following lemma determines the sign of the function $\sigma(t) + \mu(t')\,\delta\,|x(t)|$ at time instants following t' provided $\sigma'(t') = 0$ and $|\mu(t')| = 1$.

Lemma 7.4-2. If (7.149) is met with some constant $c > 0$ and $\delta > 0$ and for a V solution of the $S_{\mu\rho}$ system (7.111)-(7.112) $\sigma'(t') = 0$, $|\mu(t')| = 1$, and $|x(t')| \neq 0$ at some t', then for t posterior and close to t'

$$\mu(t)[\sigma(t) + \delta\mu(t') |x(t)|] > 0 \tag{7.151}$$

Proof. Let for convenience $\mu(t') = 1$. Then for $t \geqslant t'$ the function $\sigma(t) + \delta |x(t)|$ may be described by the equation

$$\begin{aligned}\sigma(t)+\delta|x(t)| &= \int_{t'}^{t}\{\sigma[x(\tau)+\delta|x(\tau)|\}^{\cdot}\,d\tau \\ &= \int_{t'}^{t} c\left[\left(a_1 - a_2\frac{\xi_\lambda}{c} + \frac{\xi_\lambda\xi_\mu}{c^2}\right)x + k^0|x|\right]d\tau\end{aligned} \tag{7.152}$$

where as before $\xi_\lambda = 1 + \delta\lambda \operatorname{sgn} x$, $\xi_\mu = 1 - \delta \operatorname{sgn} x$, and $\lambda = \sigma/\delta|x|$.

Now we note that under the conditions of the lemma and the assumption of $\lim_{t\to t'} \lambda(t) = -1$, once the relationship (7.149) holds, the integrand in (7.152) is positive. Consequently, for t posterior and close to t' $\sigma(t) + \delta|x(t)| > 0$, which coincides with (7.151). Q.E.D.

The last lemma in the series yields, under certain conditions, a bound for $\sigma(t)$ over some time interval.

Lemma 7.4-3. If (7.149) is met with some constant $c > 0$ and $\delta > 0$ and for some V solution $[x(t), \mu(t)]$ of the $S_{\mu\rho}$ system (7.111)-(7.112) in $[t_1, t_2]$ $|\sigma(t_1)| < \delta|x(t_1)|$, $|\sigma'(t)| \leqslant |x(t)|\delta/2$, then for all $t \in [t_1, t_2]$

$$|\sigma(t)| < \delta|x(t)| \tag{7.153}$$

Proof. Assume the converse, i.e. $|\sigma(t')| = \delta|x(t')|$ at some $t' \in (t_1, t_2]$ and $|\sigma(t)| < \delta|x(t)|$ for $t \in [t_1, t')$. Without any loss of generality we may assume here that $\sigma(t') < 0$. Then

$$\sigma(t') + \delta|x(t')| = 0 \tag{7.154}$$

and for t close to t' with $t < t'$

$$\sigma(t) + \delta|x(t)| > 0 \tag{7.155}$$

We notice now that by virtue of the lemma condition $|\sigma'(t)| \leqslant |x(t)|\delta/2$ for all $t \in [t_1, t_2]$ and the equality (7.154) we have $\sigma'(t') \leqslant 0$, therefore $\sigma(t') + \delta\mu(t')|x(t')| + |x(t')|\delta/2 \geqslant 0$. Resolving this inequality for $\mu(t')$ we find with account of (7.154) that $\mu(t') \geqslant 1/2$. Let us now put down the equation for the function

$\sigma(t) + \delta \mid x(t) \mid$ for $t < t'$ as derived with $S_{\mu\rho}$ system's equations of motion, like we did in (7.152), viz.,

$$\sigma(t)+\delta \mid x(t) \mid = \int_t^{t'} c\left[\left(a_1-a_2\frac{\xi_\lambda}{c}+\frac{\xi_\lambda\xi_\mu}{c^2}\right) x+\mu k^0 \mid x \mid\right] d\tau \quad (7.156)$$

where $\xi_\lambda = 1 + \delta\lambda \operatorname{sgn} x$, $\xi_\mu = 1 - \delta_\mu \operatorname{sgn} x$, and $\lambda = \sigma/\delta \mid x \mid$.

Using (7.149) and the proved inequality $\mu(t') \geqslant 1/2$ in the integrand we see that this expression is nonnegative and as $t < t'$ we immediately arrive at $\sigma(t) + \delta \mid x(t) \mid \leqslant 0$ for $t < t'$. This, however, contradicts (7.155), therefore no such instant $t' \in [t_1, t_2]$ exists. Q.E.D. Now we are in a position to prove the theorem.

Proof of the theorem. Recall that we consider a V' solution $[x(t), \mu(t)]$ for $t \geqslant t_0$ and assume that there exist time instants such that $\mid \mu(t) \mid = 1$. Let t_1 be a first such time instant. It is clear that $t_1 > t_0$ and for $t \in [t_0, t_1)$ $\mid \mu(t) \mid < 1$. By Lemma 7.4-1 $\mu(t_1) \times \sigma'(t_1) \leqslant 0$. For definiteness we shall assume further that $\mu(t_1) = 1$, so that $\sigma'(t_1) \leqslant 0$. If for $t \in [t_1, t_2]$ $\sigma' < 0$, then in view of (7.111) $\mu(t) = 1$ for such t and hence $[t_1, t_r]$ is the desired time interval of constant μ.

Let now t_2 be a time instant such that $\sigma'(t_2) = 0$ and $\mu(t) = 1$ for $t \in [t_1, t_2]$. By Lemma 7.4-2 for t close to t_2, $t > t_2$, $\sigma(t) + \delta \mid x(t) \mid > 0$, and therefore $\mid \sigma(t) \mid < \delta \mid x(t) \mid$ for such t. Given (7.128), (7.129) and $\sigma'(t_2) = 0$ we get from Theorem 7.4-2 that for all t in $[t_2, t_r]$ $\mid \sigma'(t) \mid \leqslant \mid x(t) \mid \delta/2$. Applying Lemma 7.4-3 to the two last inequalities yields $\mid \sigma(t) \mid < \delta \mid x(t) \mid$ for all $t \in (t_2, t_r]$. This, however, implies that $\mid \mu(t) \mid < 1$ for $t \in (t_2, t_r]$. To prove, assume the converse holds, i.e. $\mu(t') = 1$ for some $t' \in (t_2, t_r]$. This, however, entails $\sigma'(t') = \sigma(t') + \delta \mid x(t') \mid \leqslant 0$, hence $\sigma(t') \leqslant -\delta \mid x(t') \mid$, which contradicts the proved inequality $\mid \sigma(t) \mid < \delta \mid x(t) \mid$ for all $t \in (t_2, t_r]$. Consequently, the interval $[t_1\ t_2]$ is the only one in $[t_0, t_r]$ to which Theorem 7.4-4 points. This completes the proof of the theorem.

In summarizing, we have demonstrated that under an additional condition such as (7.149) the $\mathcal{A}_{\mu\rho}$ control algorithm (7.111) examined is indeed quasicontinuous. The cost for obtaining such property is that the control problem can be solved Δ-accurately only.

7.5 $S_{\mu\rho}$ (2) Systems with Integral Coordinate-Operator Feedback

This section is a qualitative investigation of a binary control system resulted from incorporating a static OFB loop into the $S_{\mu\rho}$ (2) system of Section 6.11. We shall aim here at establishing relationships enabling parametric synthesis of a closed-loop control system with desired properties. Another goal will be to reveal new

features appearing upon the incorporation of an OFB loop. We shall demonstrate that the class of $\mathcal{F}_x$-processes controlled in this way can be extended by denying the assumption (7.67).

A salient feature of $S_{\mu\rho}$ systems considered in this section is associated with the fact that the CFB law explicitly involves $x(t)$ and $\dot{x}(t)$ [see (6.116)]. This implies that a static OFB law to be employed must be different from (7.13). Indeed it seems natural, though not necessary, to alter the dynamic properties of the reference input element R_x in the block diagram of Fig. 7.1 in such a way as to change the coefficients affecting both $x(t)$ and $\dot{x}(t)$. It is quite obvious that such arrangement of operator feedback provides greater capabilities, as compared with the law (7.13), for getting closer dynamic performance for the reference input element R_x and the S_μ system proper.

More specifically this method of incorporating an OFB loop means that the reference signal of the element should be formed as

$$x^r(t) = -[c - \rho_2(t)]\,\dot{x}(t) + \rho_1(t)\,x(t) \tag{7.157}$$

where ρ_1 and ρ_2 are OFB loop output signals.

In vector matrix form, where $\boldsymbol{x} = [x, \dot{x}]^T$, $\boldsymbol{c} = [1, c]$, and

$$\boldsymbol{\rho} = [\rho_1, \rho_2] \tag{7.158}$$

equation (7.157) rewrites as

$$x^r(t) = -\boldsymbol{c}\boldsymbol{x}(t) + \boldsymbol{\rho}(t)\,\boldsymbol{x}(t) + x(t) \tag{7.159}$$

With this method of operator feedback incorporation the COFB loop error takes the form

$$\sigma'(t) = x^r(t) - x(t) = -\boldsymbol{c}\boldsymbol{x} + \boldsymbol{\rho}(t)\,\boldsymbol{x}(t) \tag{7.160}$$

This equation suggests a natural way of generating a vector-function $\boldsymbol{\rho}(t)$. In order to secure, as usual, $|\sigma'(t)| \leqslant \delta\,|x(t)|$ for some constant $\delta > 0$, the components $\rho_1(t)$ and $\rho_2(t)$ can be chosen as

$$\begin{aligned} \rho_1(t) &= \delta\beta_1\mu(t)\,\operatorname{sgn} x(t) \\ \rho_2(t) &= \delta\beta_2\mu(t)\,\operatorname{sgn}\dot{x}(t) \end{aligned} \tag{7.161}$$

where β_1 and β_2 are some numbers, $0 \leqslant \beta_i \leqslant 1$, $i = 1, 2$.

Note that for $\beta_2 = 0$ this approach coincides with the method used earlier [see (7.13) and Fig. 7.3]. With both β's other than zero this operator feedback is a vector feedback.

For convenience of our further discussion we introduce in addition

to the row 2-vector $\boldsymbol{\beta} = [\beta_1, \beta_2]$ the diagonal matrix

$$S^x = \begin{bmatrix} \operatorname{sgn} x & 0 \\ 0 & \operatorname{sgn} \dot{x} \end{bmatrix}$$

and a column vector $|\boldsymbol{x}| = |x|, |\dot{x}|^{\mathrm{T}}$.

In this notation $S^x \boldsymbol{x} = |\boldsymbol{x}|$, the operator feedback introduced in this section is described as

$$\boldsymbol{\rho}(t) = \delta\mu(t)\,\boldsymbol{\beta} S^x \tag{7.162}$$

and the COFB loop error as

$$\sigma'(t) = \sigma(t) + \delta\mu(t)\,\boldsymbol{\beta}\,|x(t)| \tag{7.163}$$

where as earlier $\sigma(t) = \sigma[\boldsymbol{x}(t)] = -\boldsymbol{c}\boldsymbol{x}(t)$.

The above remarks and the way to introduce an OFB loop completely define the form of $\mathscr{A}_{\mu\rho}$ (2) algorithm used in this section, namely, $u(t) = \mu(t)\,\boldsymbol{k}^0\,|\boldsymbol{x}(t)|$

$$\begin{aligned} \dot{\mu}(t) &= -\alpha \operatorname{sgn} \sigma'(t) \quad \text{for} \quad |\mu(t)| \leqslant 1 \\ &= -\omega\mu(t) \quad \text{for} \quad |\mu(t)| > 1,\ |\mu(t_0)| \leqslant 1,\ t \geqslant t_0 \end{aligned} \tag{7.164}$$

$$\sigma'(t) = \sigma(t) + \delta\mu(t)\,\boldsymbol{\beta}\,|\boldsymbol{x}(t)| \quad \sigma(t) = -\boldsymbol{c}\boldsymbol{x}(t)$$

where $\boldsymbol{k}^0 = [k_1^0, k_2^0]$ is the row vector of positive components k_i^0, $i = 1, 2$.

These equations taken together with the equation of $\mathscr{P}_x$-process

$$\ddot{x}(t) = a_2(t)\,\dot{x}(t) + a_1(t)\,x(t) = -u(t) \tag{7.165}$$

where $a_1(t)$ and $a_2(t)$ are within (4.2), describe the motion of the $S_{\mu\rho}$ (2) system being examined.

The equation in the operator variable suggests that our analysis of the $S_{\mu\rho}$ (2) system will be limited to the evaluation of its V solutions $[\boldsymbol{x}(t), \mu(t)]$. As in other $S_{\mu\rho}$ systems, the major goal of this study is to evaluate conditions securing the identity $\sigma'(t) = 0$ for all $t \geqslant t_1 \geqslant t_0$. Once this equality takes place, then for the said t we have from $\sigma'[\boldsymbol{x}(t), \mu(t)]$ for the respective V solutions

$$|\sigma(t)| \leqslant \delta\boldsymbol{\beta}\,|\boldsymbol{x}(t)| \tag{7.166}$$

This inequality is equivalent to $\boldsymbol{x}(t) \in \bar{G}_\delta$, where the domain $\bar{G}_\delta$ defined in (6.120) has the form

$$\bar{G}_\delta = \{\boldsymbol{x}: |\sigma(\boldsymbol{x})| \leqslant \delta\boldsymbol{\beta}\,|\boldsymbol{x}|\} \tag{7.167}$$

In other words, if $\sigma'(t) = 0$ for some V solution $[\boldsymbol{x}(t), \mu(t)]$ of the $S_{\mu\rho}$ (2) system (7.164)-(7.165) on the semiaxis $t \geqslant t_1$, then

convergence to the origin O_x for the respective controlled process $\boldsymbol{x}(t)$ (subject to the respective estimates for its component norms) is governed by the relationships formulated in Theorem 6.11-1 and Corollary 6.11-1. The associated verbal statements can be readily deduced from the formulations in the said theorem and corollary and therefore omitted here. The foregoing discussion enables solving Problem A.

We turn to conditions ensuring $\sigma'(t) = 0$ at some $t = t_1$ for certain V solutions of the $S_{\mu\rho}$ (2) system under investigation. Evaluation of such conditions makes up the subject of Problem C.

Problem C. Consider a set Γ_β in the space $(\boldsymbol{x}, \mu)$

$$\Gamma_\beta = \{(\boldsymbol{x}, \mu): \sigma(\boldsymbol{x}) + \delta\mu\boldsymbol{\beta} \mid \boldsymbol{x} \mid = 0, \quad \mid \mu \mid \leqslant 1\} \tag{7.168}$$

Now for a V solution $\sigma'(\boldsymbol{x}, \mu) = 0$ will mean that $V \in \Gamma_\beta$. Therefore our problem reduces to evaluating whether or not Γ_β is a domain of attraction for the V solutions of the $S_{\mu\rho}$ system. Solving this problem is similar to that of attraction of the set Γ (7.18) for the $S_{\mu\rho}$ (1) system. A theorem analogous to Theorem 7.1-1 may be formulated in this connection too. Prior to embark on its formulation we demonstrate the relation between the $S_{\mu\rho}$ (2) system on hand and the VS (2) system described by

$$\ddot{x}(t) + a_2(t)\,\dot{x}(t) + a_1(t)\,x(t) = \boldsymbol{k}^0 \mid \boldsymbol{x}(t) \mid \operatorname{sgn}\sigma(\boldsymbol{x}(t)), \quad t \geqslant t_0 \tag{7.169}$$

If $\sigma'(t)$ is other than zero on a time interval $2/\alpha$ long, then at the end of the interval $\mu(t) = -\operatorname{sgn}\sigma'(t)$. Since in this case $\sigma'(t) = \sigma(t) - \delta\mu\boldsymbol{\beta} \mid \boldsymbol{x}(t) \mid \operatorname{sgn}\sigma'(t)$, the signs of $\sigma'(t)$ and $\sigma(t)$ will be the same and, consequently, equation (7.165) coincides with (7.169). In other words, the considered process $\boldsymbol{x}(t)$ satisfies the equation of VS (2) system. For this system, as will be recalled, Theorem 6.11-3 is valid. It establishes the conditions (6.134) of the domain $G_0 = \{\boldsymbol{x}: \sigma(\boldsymbol{x}) = 0\}$ being a domain of Θ-attraction (see Definition 6.7-3). Now, securing this property for the VS (2) system turns out sufficient for the set Γ_β to be attracting for the $S_{\mu\rho}$ (2) system. This fact receives a substantial formulation below.

Theorem 7.5-1. If with some positive constants c and δ

$$k_i^0 > \bar{k}_i^0, \quad i = 1, 2 \tag{7.170}$$

where $\bar{k}_i^0$, $i = 1, 2$, are defined by (6.123b), then the set Γ_β, defined by (7.168) is a domain of attraction for the $S_{\mu\rho}$ (2) system (7.164)-(7.165).

An elaborate proof for this theorem parallels that for Theorem 7.7-1 therefore we content ourselves with only a brief comment.

Suppose the converse is true, i.e. that $V \notin \Gamma_\beta$ for all $t \geqslant t_0$. Then we find that $\sigma'(t)$ do not vanish for this V solution. In the circumstances, the associated process $x(t)$ for $t \geqslant t_0 + 2/\alpha$ satisfies the VS (2) system equation (7.169). By Theorem 6.11-3 for this solution there is a time instant t_1 such that either $\sigma(x(t_1)) = 0$ or no such instant exists but $\lim_{t\to\infty} \| x(t) \| = 0$.

For the first case, analysis of $\sigma'(t) = \sigma(t) - \delta\beta \mid x(t) \mid \operatorname{sgn} \sigma'(t)$ valid for $t \geqslant t_1$ we find that at $t = t_1$ $\sigma'(t)$ must be zero, which contradicts the assumption. Hence at some $t < t_1$ $\sigma'(t)$ has to vanish. For the second case, we have $\lim_{t\to\infty} V \in \Gamma_\beta$. Hence Γ_β is attracting for the V solutions of the $S_{\mu\rho}$ (2) system.

This theorem indicates in particular that all controlled processes in the examined system divide themselves into two families when the conditions (7.170) are met, namely, $x^{\mathrm{I}}(t)$ processes for which the said time t_1 does exist, and $x^{\mathrm{II}}(t)$ processes for which $\lim_{t\to\infty} \| x^{\mathrm{II}}(t) \| = 0$. This remark essentially terminates the investigation of x^{II} processes. The behavior of x^{I} processes for $t \geqslant t_1$ will be evaluated below in Problem B.

Problem B. Recall that this problem is to evaluate conditions ensuring $x(t) \in \overline{G}_\delta$ for all $t \geqslant t_1$ if for $t = t_1$ $x(t_1) \in \overline{G}_\delta$. These conditions are determined in the following theorem.

Theorem 7.5-2. If with some constants $c > 0$ and $\delta > 0$ the inequalities (6.125) and (7.170) are met, and

$$\alpha \geqslant \max_{i=1,\,2;\; c\in\Sigma_2} (k_i^0 + \overline{\overline{k}}_i^0)\, \tilde{c}b/\delta\beta_i \tag{7.171}$$

where the set Σ_2 is defined as

$$\Sigma_2 = \{\tilde{c}:\ \tilde{c} = c + \lambda\xi,\ \ |\lambda| \leqslant 1,\ \ \xi = [\xi_1, \xi_2],\ \ \xi_i = \pm\delta\beta_i,\ \ i = 1, 2$$

and the numbers $\overline{\overline{k}}_i^0$, $i = 1, 2$, are defined in (6.123c). Then the set Γ_β (7.168) is invariant on V solutions of the $S_{\mu\rho}$ (2) system (7.164)-(7.165).

Comparing (7.170), (7.171) for the $S_{\mu\rho}$ (2) system with (6.129)-(6.130) for the S_μ (2) system indicates that the vector OFB (7.162) enables us to keep intact the CFB loop gains and to halve the COFB loop gain. In other words, the desired inequality $|\sigma| \leqslant \delta\beta |x|$ is ensured with smaller (in the COFB loop) gain parameters, i.e. operator feedback does draw closer the dynamic behavior of the COFB loop reference input element to the dynamic properties of the S_μ (2) system.

A comment on the proof. Here we again limit ourselves to giving but the major points essential for the proof. Details may be recovered from the proof of Theorem 7.1-2.

In vector matrix form, equation (7.165) subject to the control from (7.164) becomes

$$\dot{x}(t) = A(t)\,x(t) - \mu(t)\,bk^0 \mid x(t) \mid, \quad t \geqslant t_0$$

where $A(t) = [a^1(t), a^2(t)]$ is the 2×2 matrix of the columns

$$a^1(t) = \begin{bmatrix} 0 \\ -a_1(t) \end{bmatrix}, \quad a_2(t) = \begin{bmatrix} 1 \\ -a_2(t) \end{bmatrix}, \text{ and where } b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

With this notation the expression for $\dot{\sigma}'(t)$ may be rewritten, with account of $\sigma'(t_0) = 0$, for $t \geqslant t_0$ as

$$\begin{aligned} \sigma'(t) &= \int_{t_0}^{t} [\sigma(\tau) + \delta\mu(\tau)\,\beta \mid x(\tau) \mid]^{\cdot}\, d\tau \\ &= \int_{t_0}^{t} [-\tilde{c}Ax + \tilde{c}bk^0 \mid x \mid \mu + \delta\dot{\mu}\beta \mid x \mid]\, d\tau \end{aligned} \tag{7.172}$$

where $\tilde{c} = c - \delta\mu\beta S^x$.

At $\mid \mu \mid = 1$ equation (7.172) rewrites

$$\begin{aligned} \sigma'(t) = \int_{t_0}^{t} [&(-\tilde{c}a^1 x + \tilde{c}bk_1^0 \mid x \mid \operatorname{sgn}\mu \\ &+ (-\tilde{c}a^2\dot{x} + \tilde{c}bk_2^0 \mid \dot{x} \mid \operatorname{sgn}\mu)]\, d\tau \end{aligned} \tag{7.173}$$

whereas for $\mid \mu \mid < 1$ (7.172) and (7.173) yield

$$\sigma'(t) = \int_{t_0}^{t} [-\tilde{c}Ax + \tilde{c}bk^0 \mid x \mid \mu - \alpha\delta\beta \mid x \mid \operatorname{sgn}\sigma']\, d\tau \tag{7.174}$$

Examine the integrands in (7.172) and (7.164) assuming $\| x(t) \| > 0$ for $t \geqslant t_0$ with account of the fact that the product cb is strictly positive provided (6.125) is satisfied. Our reasoning will follow along the lines similar to that used in Theorem 7.1-2. As a result we establish that $\sigma'(t) = 0$ for all $t \geqslant t_0$ given (7.170) is satisfied in the first case, and subject to

$$\alpha \geqslant (-\tilde{c}Ax + \tilde{c}bk^0 \mid x \mid \mu)/\delta\beta \mid x \mid \tag{7.175}$$

for all $t \geqslant t_0$ in the second case.

Observing now that the maximum of the right-hand side in (7.175) never exceeds the largest ratio of the moduli of the coefficients of $\mid x(t) \mid$ and $\mid \dot{x}(t) \mid$ in the numerator and denominator, we find out that this inequality holds whenever (7.171) is valid. Therefore, in the conditions of Theorem 7.5-2 $\sigma'(t) = 0$ for all $t \geqslant t_0$ if $\sigma'(t_0) = 0$, which means the invariance of the set Γ_β on V solutions of the S_μ (2) system (7.174)-(7.175). These remarks close our comment on the proof.

Notice now that all the relationships defining attraction and invariance of Γ_β and convergence to the origin O_x for those processes which from some time instants belong to $\overline{G}_\delta$ and $x(t) \notin \overline{G}_\delta$ can be satisfied by a suitable choice of system's circuit parameters. Therefore, the above findings establish $\mathcal{A}_{\mu\rho}(2)$ controllability of the $\mathcal{P}_x$-process with the algorithms (7.164). That $\mathcal{P}_x$-processes are steady $\mathcal{A}_{\mu\rho}(2)$ controlled in this case can be proved in the same way as we did for the $S_\mu(1)$ system of Section 6.7. In shorthand our findings can be formulated as follows:

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_{\mu\rho}(2)} \left\{ \mathcal{F}_c,\ I_2;\ \frac{c_m}{\bar{\delta}} \left(\frac{1+c+\bar{\delta}}{c} \right)^2 \exp(2P/\alpha) \right\} \tag{7.176}$$

where $\bar{\delta} = \delta\beta_m(c+1)/(c-\delta\beta_m)$ with $\beta_m = \max(\beta_1, \beta_2)$, P is defined by (6.137) as

$$P = \max_{i=1,\,2;\ |\alpha|\leqslant 1} \sup_{t\geqslant t_0} \| \boldsymbol{a}^i(t) - \alpha \boldsymbol{b} k_i^0 \|$$

and α must satisfy (7.171).

To conclude our investigation of $S_{\mu\rho}(2)$ systems (7.164)-(7.165) we need only to clarify some features in the control process dependence on the $\mathcal{P}_x$ parameters $a_1(t)$ and $a_2(t)$.

Problem F. Henceforth we drop the superscript in the x^{I} process designation as only this family of processes will be examined. Suppose all the conditions are present which warrant $\sigma'(t) = \sigma(t) + \delta\mu(t)\beta |x(t)| = 0$ beginning from a time t_1. Resolving this equation for $\mu(t)$ subject to $\|x(t)\| > 0$ and substituting the result $\mu(t) = -\sigma(t)/\delta\beta |x(t)|$ in the expression for control yields

$$u(t) = -\frac{1}{\delta} \frac{k^0 |x(t)|}{\beta |x(t)|} \sigma(t), \quad t \geqslant t_1 \tag{7.177}$$

Denote $k_m^0 = \max(k_1^0, k_2^0)$, and define β_i as

$$\beta_i = k_i^0/k_m^0, \quad i = 1, 2 \tag{7.178}$$

With this choice of the parameters the control signal $u(t)$ of (7.177) becomes

$$u(t) = -k_m^0 \sigma(t)/\delta = k_m^0/\delta [c\dot{x}(t) + x(t)] \tag{7.179}$$

Substituting this expression into (7.165) yields the linear uniform differential equation for $t \geqslant t_1$

$$\ddot{x}(t) + [a_2(t) + ck_m^0/\delta] \dot{x}(t) + [a_1(t) + k_m^0/\delta] x(t) = 0 \tag{7.180}$$

Again, as in Section 7.1, we see that the nonlinear control system ($S_\mu(2)$ system of Section 6.11) upon incorporation of OFB can be described on a certain domain $\overline{G}_\delta \subset \mathbb{R}_x^2$ by a linear equation such as (7.180). The nonlinear feedback itself becomes equivalent to the linear feedback (7.179) with constant parameters, a sufficiently

small δ helps realizing the effect of a large gain whose role is played by k_m^0/δ. It should be noted that direct use of the control law (7.179) is impossible because the constraint imposed on the class of admissible controls [see (6.1a)] no longer holds.

Equation (7.180) is seen to coincide with (7.26) accurate to the designation of one coefficient. Therefore, for the dependence of $x(t)$ considered in the $S_{\mu\rho}$ (2) system, in general, the same conclusions apply which have been drawn in Section 7.1 for processes in the $S_{\mu\rho}$ (1) system. Some difference exists in the estimates concerning the extent of parametrically static behavior.

For the $S_{\mu\rho}$ system under investigation the function $\lambda_\beta(t) = \sigma(t)/\delta\beta \mid x(t) \mid$, with a_1 and a_2 within (4.2), tends to the limit

$$\lambda_\beta^* = -\frac{1+cs^*}{\delta\beta_1+\delta\beta_2 \mid s_m^* \mid} \operatorname{sgn} x$$

where s_m^* is the largest root of the algebraic equation (7.27) at $\tilde{k}^0 = k_m^0$, generally different from the number $\lambda^* = -(1 - cs^*) \operatorname{sgn} x/\delta$, found in Section 7.1 for the $S_{\mu\rho}$ (1) system.

Because the equation $\sigma'(t) = 0$ can be represented as

$$(c+\delta\beta_2\lambda_\beta \operatorname{sgn} \dot{x})\ddot{x}+(1+\delta\beta_1\lambda_\beta \operatorname{sgn} x)\dot{x}=0$$

in the state space to this equation there corresponds the nonstationary point $\pi_\beta^t = [c + \delta\beta_2\lambda_\beta \operatorname{sgn} \dot{x},\ 1 + \delta\beta_1\lambda_\beta \operatorname{sgn} x]$ which in view of the above circumstances tends to the stationary limit $\pi_\beta^* = [c + \delta\beta_2\lambda_\beta^* \operatorname{sgn} \dot{x},\ 1 + \delta\beta_1\lambda_\beta^* \operatorname{sgn} x]$. Two facts are worth mentioning in this context. First, this limit is independent of the sign of either x or $\dot{x}$; second, the margin of static behavior $\Delta_\beta^* = \| \pi_\beta^* - \pi^0 \|$ where $\pi^0 = [c, 1]$ is given as

$$\Delta_\beta^* = \frac{(\beta_1+\beta_2)|1+cs_m^*|}{\beta_1+\beta_2 \mid s_m^* \mid}$$

Letting $k^0 = k_m^0$ (in this case $s^* = s_m^*$) and comparing Δ_β^* with Δ_π^* of Section 7.1 we see that with the same δ the magnitude of the parametrically static margin in the $S_{\mu\rho}$ (2) system may be above, below, or the same as that in the $S_{\mu\rho}$ (1) system and this is completely decided by the value of $| s_m^* |$. For variable $\mathcal{P}_x$ parameters, parametrically static behavior transforms into parametrically dissipative being bounded as follows:

$$\Delta_\beta^t = \| \pi_\beta^t - \pi^0 \| \leqslant \delta(\beta_1+\beta_2) \leqslant \delta(1+\beta_{min}) \qquad (7.181)$$

where $\beta_{min} = \min(\beta_1, \beta_2)$.

This implies that for the same value of δ the range of parametrically dissipative behavior in $S_{\mu\rho}$ (2) systems is generally larger

than that in $S_{\mu\rho}$ (1) systems, for which $\Delta_{\pi}^{t} \leqslant \delta$ by (7.18). In this case, however, it falls off to zero as $\delta \to 0$, but in the limit the $S_{\mu\rho}$ (2) system in question becomes a VS (2) system, so there is no achieving a parametrically astatic behavior in this case, too.

7.6 *S* System Control with $\mathcal{A}_{\mu\rho}$ Algorithms

This section discusses binary control systems resultant from $S_{\mu\rho}$ systems of Section 6.12 when the latter incorporate a static operator feedback loop. The aim of this investigation is to generalize the findings of the foregoing sections of this chapter for the case of *S* systems. It will establish the principal relationships for design of closed-loop control systems with desired properties and demonstrate the salient features in control of *S* systems and $\mathcal{P}_x$-processes.

As will be recalled, *S* systems are described by a set of first order differential equations such as (6.139) with the constraints on system parameters (6.140), or, which is the same, by one vector equation (6.141) subject to (6.140). Propositions 6.12-1 to 6.12-3 defining the class of systems examined, carry over to these *S* systems. In accord with Definition 6.12-1 we write in this case $S \in \mathcal{S}(\delta_0)$.

Because we shall be concerned basically with S_{μ} systems of Section 6.12, essentially S_{μ} (1) systems, we may safely content ourselves with considering scalar static operator feedback. Similar to our approach in Section 7.1, such OFB can be adopted in the form

$$\rho(t) = \delta\mu(t) \operatorname{sgn} x_1(t), \quad t \geqslant t_0 \tag{7.182}$$

where as usual δ is a positive constant, $\mu(t)$ is the manipulated variable generated by the COFB loop, and $x_1(t)$ is the first component of the controlled process $\boldsymbol{x}(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$.

The signal $\rho(t)$ generated in this way will be employed to alter the parameters of the reference input element R^x for dynamic properties as

$$\boldsymbol{c}(\rho) = \boldsymbol{c} + [\rho, 0] \tag{7.183}$$

where the row vector $\boldsymbol{c} = [c', 1]$ defines the setpoint parameters in the S_{μ} system, and the row vector $\boldsymbol{c}(\rho)$ the parameters of such a reference input element in the $S_{\mu\rho}$ system.

With an operator feedback loop introduced in this way, the COFB loop error $\sigma(\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{x}$ in the S_{μ} system transforms into the error $\sigma'(\boldsymbol{x})$ in the $S_{\mu\rho}$ system. Equations (7.182)-(7.183) yield

$$\sigma'\boldsymbol{x}(t) = \sigma[\boldsymbol{x}(t)] + \delta\mu(t) \mid x_1(t) \mid, \quad t \geqslant t_0 \tag{7.184}$$

The $\mathcal{A}_{\mu\rho}$ algorithm to be examined has the form

$$u(t) = \mu(t) k_1^0 \mid x_1(t) \mid$$

$$\dot{\mu}(t) = -\alpha [\mu(t) + \operatorname{sgn} \sigma'(t)], \quad \mid \mu(t_0) \mid \leqslant 1, \quad t \geqslant t_0$$

$$\sigma'(t) = \sigma(t) + \delta\mu(t) \mid x_1(t) \mid \tag{7.185}$$

$$\sigma(t) = \boldsymbol{c}\boldsymbol{x}(t)$$

where k_1^0 and α are positive constants, $\boldsymbol{c} = [c', 1]$, $c' = \text{const} > 0$, all being circuit parameters of the control algorithm.

This set of equations augmented by the S system equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t) u(t), \quad t \geqslant t_0 \tag{7.186}$$

completely describes the $S_{\mu\rho}$ system, i.e. defines its equations of motion. Here $\boldsymbol{A}(t) = [\boldsymbol{a}^1(t), \boldsymbol{a}^2(t)]$ with $\boldsymbol{a}^1(t) = [a_{11}(t), a_{21}(t)]^{\mathrm{T}}$ and $\boldsymbol{a}^2(t) = [a_{21}(t), a_{22}(t)]^{\mathrm{T}}$, and $\boldsymbol{b}(t) = [b_1(t), b_2(t)]^{\mathrm{T}}$. The components of these vectors are piecewise continuous functions of time, for all $t \geqslant t_0$ subject to the range constraints

$$a_{ij}^- \leqslant a_{ij}(t) \leqslant a_{ij}^+; \quad i, j = 1, 2$$

$$b_j^- \leqslant b_j(t) \leqslant b_j^+; \quad j = 1, 2 \tag{7.187}$$

where all time functions are deemed unknown, whereas all constants are known and incorporated into the description of the set I_2, characterizing information on the S system available for control, viz.

$$I_2 = \{\boldsymbol{x}(t), a_{ij}^{\pm} (i, j = 1, 2), b_j^{\pm} (j = 1, 2), \quad t \geqslant t_0\}$$

Solutions of the $S_{\mu\rho}$ system are seen, therefore, to consist of functional pairs $[\boldsymbol{x}(t), \mu(t)]$ among which the initial condition $\mid \mu(t_0) \mid \leqslant 1$ selects V solutions. We shall also consider V' solutions for which $\mid \mu(t_0) \mid < 1$.

In this case the choice of parameters is also aimed at securing the equality $\sigma'(t) = 0$ beginning from some time instant $t_1 \geqslant t_0$. Then for V solutions the controlled process $\boldsymbol{x}(t)$ will belong to G_δ for $t \geqslant t_1$, where

$$G_\delta = \{x: \mid \sigma(x) \mid \leqslant \delta \mid x_1 \mid\} \tag{7.188}$$

The asymptotic behavior of $\boldsymbol{x}(t) \in G_\delta$ was examined in Problem A of Section 6.12. The respective solution is formulated in Theorem 6.12-1 and Corollary 6.12-1. Therefore we go over to evaluating conditions retaining $\boldsymbol{x}(t)$ in G_δ for $t \geqslant t_1$ if $\boldsymbol{x}(t_1) \in G_\delta$. A definition of the respective conditions makes up Problem B.

Problem B. Define the set

$$\Gamma_\delta = \{(x, \mu): \sigma(x) + \delta\mu \mid x_1 \mid = 0, \mid \mu \mid \leqslant 1\} \tag{7.189}$$

Again we check that $V \in \Gamma_\delta$ means $\boldsymbol{x}(t) \in G_\delta$. The solution is given by the following theorem.

Theorem 7.6-1. Let h be a positive number, $0 < h < 1$, and at some constant $\boldsymbol{c}$ and δ $S \in \mathcal{S}(\delta_0)$. Suppose that

$$(1 - h)\, k_1^0 = \bar{k}_1^0 \tag{7.190}$$

where $\bar{k}_1^0$ is given by (6.158), and

$$\alpha \geqslant \max_{\tilde{c} \in \Sigma_2} \sup_{t \geqslant t_0} (k_1^0 + \bar{\bar{k}}_1^0)\, \tilde{\boldsymbol{c}}\boldsymbol{b}/\delta \tag{7.191}$$

where $\bar{\bar{k}}_1^0$ is defined in (6.160) and vector $\boldsymbol{c}$ in the Σ_2' set definition of (6.157b). Then the set Γ_δ defined in (7.189) is invariant on V solutions of the $S_{\mu\rho}$ system (7.185)-(7.187).

Comparison of (7.190)-(7.191) with (6.161), (6.162) for the S_μ system indicates that the incorporation of an OFB loop (7.182)-(7.183) into such a S_μ system is helpful in that the gain parameters in the CFB and COFB loops can be substantially diminished and a tradeoff in their choice eliminated.

Proof. Consider for $t \geqslant t_0$ a V solution $[x(t), \mu(t)]$ of the $S_{\mu\rho}$ system (7.185)-(7.187) belonging to Γ_δ at $t = t_0$. For this solution $\sigma'(t_0) = 0$. We are to demonstrate that given (7.190) and (7.191) $\sigma'(t) = 0$ is identically zero for all $t \geqslant t_0$.

In the simplest case of $x_1(t_0) = 0$ we have $\sigma(t_0) = 0$ from $\sigma'(t_0) = 0$. Then $x_2(t_0) = 0$ and hence $\| \boldsymbol{x}(t) \| = 0$ for all $t \geqslant t_0$. Now assume $| x_1(t_0) | > 0$. All V solutions with such controlled processes belong to the set Γ_δ, therefore we may safely assume $| x_1(t) | > 0$ for all $t \geqslant t_0$, so that now

$$\begin{aligned}
\sigma'(t) &= \int_{t_0}^{t} [\sigma(\tau) + \delta\mu(\tau) \mid x_1(\tau) \mid]^{\cdot}\, d\tau \\
&= \int_{t_0}^{t} \boldsymbol{cAx} + \boldsymbol{cb}\mu k_1^0 \mid x_1 \mid + \delta\mu \operatorname{sgn}(x_1\dot{x}_1) + \delta \mid x_1 \mid \dot{\mu}]\, d\tau \\
&= \int_{t_0}^{t} [\boldsymbol{ca}^1 x_1 + \boldsymbol{ca}^2 x_2 + \boldsymbol{cb}\mu k_1^0 \mid x_1 \mid + \delta\mu \boldsymbol{R}_1^x \boldsymbol{a}^1 x_1 \\
&\quad + \delta\mu \boldsymbol{R}_1^x \boldsymbol{a}^2 x_2 + \delta\mu^2 \boldsymbol{R}_1^x \boldsymbol{b} k_1^0 \mid x_1 \mid + \delta \mid x_1 \mid \dot{\mu}]\, d\tau
\end{aligned} \tag{7.192}$$

where $\boldsymbol{R}_1^x = [\operatorname{sgn} x_1, 0]$.

Eliminate $x_2(t)$ between this expression and $x_2 = \sigma - c'x_1 = \lambda\delta \mid x_1 \mid - c'x_1 = -\tilde{c}'x_1$, where $\lambda(t) = \sigma(t)/\delta \mid x_1(t) \mid$ is defined for all $t \geqslant t_0$ as $| x_1(t) | > 0$. As a result (7.192) rewrites

$$\sigma'(t) = \int_{t_0}^{t} \{[\tilde{\boldsymbol{c}}\boldsymbol{a}^1 - \tilde{c}'\tilde{\boldsymbol{c}}\boldsymbol{a}^2]\, x_1 + \tilde{\boldsymbol{c}}\boldsymbol{b}\mu k_1^0 \mid x_1 \mid + \delta \mid x_1 \mid \dot{\mu}\}\, d\tau \tag{7.193}$$

where $\tilde{\boldsymbol{c}} = \boldsymbol{c} + \delta\mu \boldsymbol{R}_1^x$.

At $| \mu(t_0) | = 1$ this expression becomes

$$\sigma'(t) = \int_{t_0}^{t} \{[\tilde{\boldsymbol{c}}\boldsymbol{a}^1 - \tilde{c}'\tilde{\boldsymbol{c}}\boldsymbol{a}^2]\, x_1 + \tilde{\boldsymbol{c}}\boldsymbol{b} k_1^0 \mid x_1 \mid \operatorname{sgn}\mu\}\, d\tau \tag{7.194}$$

whereas for $|\mu(t_0)| < 1$, observing (7.185) it can be rewritten as

$$\sigma'(t) = \int_{t_0}^{t} \{[\tilde{c}a^1 - \tilde{c}'\tilde{c}a^2]\, x_1 + \tilde{c}bk_1^0 |x_1| \mu - \alpha\delta |x_1| (\mu + \operatorname{sgn}\sigma')\}\, d\tau \tag{7.195}$$

To analyze these expressions we invoke the argument of the proof of Theorem 7.1-2 and establish that given (7.190) equation (7.194) yields $\sigma'(t) = 0$ for $t \geqslant t_0$, and with (7.191) this identity also holds. Consequently, given $\sigma'(t_0) = 0$ and the conditions of Theorem 7.6-1 are satisfied, then for all $t \geqslant t_0$ $\sigma'(t) = 0$, which is equivalent to invariance of the set Γ_δ. Q.E.D.

Thus to establish $\mathcal{A}_{\mu\rho}$ controllability of the S system it remains to prove that the set Γ_δ is a domain of attraction, therefore we go over to Problem C.

Problem C. To solve this problem we invoke the method developed in Section 6.12. In addition to the $S_{\mu\rho}$ system (7.185)-(7.187) we consider the related VS_h system described by

$$\dot{x}(t) = A(t)\, x(t) - b(t)\, k_1^h(t) |x_1(t)| \operatorname{sgn}\sigma[x(t)], \quad t \geqslant t_0 \tag{7.196}$$

where $k_1^h(t)$ is a scalar continuous function satisfying at some constant h, $0 < h < 1$, the inequality

$$k_1^h(t) > k_1^0(1 - h) \tag{7.197}$$

For the relation of controlled processes of the $S_{\mu\rho}$ system (7.185)-(7.187) to solutions of the VS_h system (7.196), all remarks made in Section 6.8 on similar relation apply here as well. Now if the domain $G_0 = \{x: \sigma(x) = 0\}$ is $\mathcal{O}$ attracting for the VS_h system (7.196) then the set Γ_δ is attracting for the $S_{\mu\rho}$ system (7.185)-(7.187). Recalling the conditions on the domain G to be $\mathcal{O}$ attracting for the VS_h system determined in Theorem 6.12-3 we combine them with some elements of the proof of Theorem 7.5-1 to verify the following statement.

Theorem 7.6-2. Let h be a positive number, $0 < h < 1$, and with some constants c and $\delta > 0$ $S \in \mathcal{S}(\delta_0)$ and

$$\sup_{t \geqslant t_0} ca^2(t) < 0 \tag{7.198}$$

$$(1 - h)\, k_1^0 > \tilde{k}_1^0 \tag{7.199}$$

where the constant $\tilde{k}_1^0$ is defined in (6.158). Then the set Γ_δ defined in (7.189) is attracting for the $S_{\mu\rho}$ system (7.185)-(7.187). All the controlled processes $x(t)$ in such a system divide themselves into two families, namely $x^{\mathrm{I}}(t)$ processes which from some time instant appear in G_δ defined in (7.188), and $x^{\mathrm{II}}(t)$ processes which do not belong to G_δ in addition $\lim_{t \to \infty} \| x^{\mathrm{II}}(t) \| = 0$ and for $x^{\mathrm{I}}(t)$ the bound (6.151) holds.

We omit the proof for this theorem to save space. Using this theorem it is not hard to understand that given $S \in \mathscr{S}(\delta_0)$ and (7.198) the control algorithm (7.185) renders the examined S system $\mathscr{A}_{\mu\rho}$ controlled. The fact that such an S system is steady $\mathscr{A}_{\mu\rho}$ controlled with this algorithm follows from the bound established in Lemma 6.12-2 and applicable in this case. Therefore this investigation may be summarized in shorthand as follows:

$$S \in \mathscr{C}_{\mathscr{A}_{\mu\rho}}\{\mathscr{F}_c, I_2; N\} \tag{7.200}$$

where the expression for N is as given in (6.176) with α which must be taken from (7.191).

The condition (7.198) confining the class of S system for which (7.200) is valid can be lifted by changing from $\mathscr{A}_{\mu\rho}$ (1) algorithms (7.185) to $\mathscr{A}_{\mu\rho}$ (2) algorithms with vector static operator feedback such as (7.162).

Problem F. In this problem our examination will be focused on x^{I} processes only and from the time when $\sigma'(t) = \sigma(t) + \delta\mu(t) \times |x_1(t)| = 0$. All the conditions of Theorems 6.12-1, 7.6-1, 7.6-2, and Corollary 6.12-1 are deemed satisfied.

From the above equation $\sigma(t) = -\delta\mu(t)\,|x_1(t)|$ for $t \geqslant t_1$ and equation (6.185) can be written as

$$\dot{y}(t) = a_c(t)\,y(t) - h_c(t)\,\delta\mu(t)\,|x_1(t)|, \quad t \geqslant t_1 \tag{7.201}$$

Here $h_c = b_c + a_c b_c' - \dot{b}_c'$ with a_c, b_c, and b_c' defined in (6.144), and the variable $y(t)$ is related to $x_1(t)$ as derived in Section 6.12, viz., $y(t) = x_1(t) - b_c'(t)\,\sigma(t)$. Substituting $\sigma(t)$ into this equation from $\sigma(t) = -\delta\mu(t)\,|x_1(t)|$ yields

$$y(t) = [1 + \delta b_c'(t)\,\mu(t)\,\operatorname{sgn} x_1(t)]\,x_1(t), \quad t \geqslant t_1 \tag{7.202}$$

Resolving this equation for $x_1(t)$ and substituting the result in (7.201) we get for $t \geqslant t_1$

$$\dot{y}(t) = \left[a_c(t) - \frac{\delta\mu(t)\,h_c(t)\operatorname{sgn} x_1(t)}{1 + \delta b_c'(t)\,\mu(t)\operatorname{sgn} x_2(t)}\right] y(t) \tag{7.203}$$

Compare this equation with equation (6.179) $\dot{y}(t) = a_c(t)\,y(t)$, $t \geqslant t_1$ describing ideal motion when $\sigma(t) = 0$. In the π_y parameter space, to this equation there corresponds the point $\pi_c = [a_c(t), 1]$ and to equation (7.203) the point

$$\pi_c^t = \left[a_c(t) - \frac{\delta\mu(t)\,h_c(t)\operatorname{sgn} x_1(t)}{1 + \delta b_c'(t)\,\mu(t)\operatorname{sgn} x_2(t)},\ 1\right]$$

The distance between these points can be bounded as

$$\Delta_c^t = \| \pi_c^t - \pi_c \| \leqslant \frac{\delta \mid h_c(t) \mid}{1 - \delta \mid b_c'(t) \mid} \leqslant \frac{\delta \sup\limits_{t \geqslant t_0} \mid h_c(t) \mid}{1 - \delta \sup\limits_{t \geqslant t_0} \mid b_c'(t) \mid} \tag{7.204}$$

This bound coincides with that of (6.188) for the S_μ system. Consequently, static operator feedback leaves the parametrically dissipative response of S_μ systems unchanged, the bounds for the magnitude of dissipative behavior for S_μ and $S_{\mu\rho}$ systems coinciding. This conclusion remains valid even under the conditions (6.182) and (6.183), the difference being that under these conditions the bound (6.188) is replaced with (6.189), and the estimate (7.204) with

$$\Delta_c^0 \leqslant \frac{\delta \mid h_c^0 \mid}{1 - \delta \mid b_1^0 / \boldsymbol{c}\boldsymbol{b}^0 \mid} \tag{7.205}$$

where the constants h_c^0 and $b_1^0/\boldsymbol{c}\boldsymbol{b}^0$ are computed by expressions given in (6.144) upon substitution of expressions (6.182) and (6.183). We see again that the bounds for Δ_c^0 and for Δ_π^0 from (6.189) coincide. The situation differs somewhat for some fixed, or even unknown, values of S system parameters $A(t) = \boldsymbol{A}$ and $\boldsymbol{b}(t) = \boldsymbol{b}$. In this case for $h_c(t) = h_c = \text{constant}$ the bound (6.188) holds for the S_μ system.

Let us derive a bound for the magnitude of parametrically dissipative response for $S_{\mu\rho}$ systems in the stated conditions. For this purpose we resolve the equation $\sigma'(t) = \sigma(t) + \delta\mu(t) \times \mid x_1(t) \mid = 0$ for $\mu(t)$ — this is legitimate as $\mid x_1(t) \mid > 0$ — to obtain

$$\mu(t) = -\sigma(t)/\delta \mid x_1(t) \mid, \quad t \geqslant t_1 \tag{7.206}$$

Now the control is given as

$$u(t) = -k_1^0 \sigma(t)/\delta = -k_1^0 \boldsymbol{c}\boldsymbol{x}(t)/\delta, \quad t \geqslant t_1 \tag{7.207}$$

Again we check that in this case all the properties of operator feedback discussed in 7.1 are present. Substituting (7.207) into (7.186) yields a linear differential equation with constant but generally unknown parameters

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - k_1^0 \boldsymbol{b}\boldsymbol{c}/\delta)\, \boldsymbol{x}(t), \quad t \geqslant t_1 \tag{7.208}$$

where $k = k_1^0/\delta > 0$ plays the role of a large gain parameter at a sufficiently small value of δ.

All the properties of solution to equation (7.208) are completely determined by the roots s_1^* and s_2^* of the characteristic equation

$$\det [sI - A + k\boldsymbol{bc}] = 0 \tag{7.209}$$

where s is the Laplace variable, and I the 2×2 unit matrix.

It can be demonstrated [26, 28] that if a pair $(A, \boldsymbol{b})$ is controllable then there exists a row vector $\boldsymbol{c}$ such that the matrix $A_k = A - k\boldsymbol{bc}$ possesses any specified spectrum (s_1^*, s_2^*) and additionally with k growing without bound one of the roots, say s_2^*, tends to $-\infty$ while the other, s_1^*, possesses a limit value coinciding with a_c corresponding to the equation of ideal motion (6.179).

With the relations for the coefficient k_1^0 given above in Theorems 7.6-1 and 7.6-2, the asymptote corresponding to s_2^* in the phase plane (x_1, x_2) always lies beyond the domain G_δ, and since $|s_1^*| < |s_2^*|$, all processes $\boldsymbol{x}(t) \in G_\delta$ as $t \to \infty$ asymptotically approach a straight line $x_2 + c^* x_1 = 0$, where c^* is a constant consistent with the "slow" root s_1^*. This constant is proportional to the ratio of the components of the eigenvector of A_k corresponding to its eigenvalue s_1^*. What should be stressed is that $\lim\limits_{k \to \infty} c^* = c'$.

This argument indicates that the variable $\mu(t)$ of (7.206) as $t \to \infty$ tends to a limit μ^* defined as

$$\mu^* = \frac{c' - c^*}{\delta} \operatorname{sgn} x_1(t_1) \tag{7.210}$$

where it is observed that $x_1(t)$ experiences no sign reversals for $\boldsymbol{x}(t) \in G_\delta$.

With this μ^* we can evaluate in the parameter space Π_y the limit point π_c^* that characterizes the ultimate differential equation to whose solution all controlled processes of $S_{\mu\rho}$ system converge. This point will be found from the expression for π_c^t upon substitution of (7.210), viz.

$$\pi_c^* = \left[a_c - \frac{(c' - c^*) h_c}{1 + (c' - c^*) b_c'}, \ 1 \right] \tag{7.211}$$

The distance from this limit (fixed) point to $\pi_c = [a_c, 1]$ corresponding to the ideal motion is given by

$$\Delta_c^* = \left| \frac{(c' - c^*) h_c}{1 + (c' - c^*) b_c'} \right| \tag{7.212}$$

Note that Δ_c^* depends on specific values of A and $\boldsymbol{b}$ in the S system and so do c^*, h_c, and h_c'. Given (6.182) and (6.183) the expression for error Δ_c^* becomes

$$\Delta_0^* = \left| \frac{(c' - c^*) h_c^0}{1 + (c' - c^*) b_1^0 / \boldsymbol{cb}^0} \right| \tag{7.213}$$

The dependence of this quantity of S system parameters is determined by the extent to which c^* depends on these parameters.

Comparing the constants (7.112), (7.113) with that of (6.188), (6.189) we see that the first always less than the last. This leads to the following qualitative conclusion. For any values of S system parameters fixed within their respective ranges of admissible variation, the incorporation of an operator feedback loop secures for the S system parametrically static behavior with a magnitude always below that of parametrically dissipative behavior for binary systems without OFB, which means a better control performance in control in such systems under constant $\boldsymbol{A}$ and $\boldsymbol{b}$.

The magnitude of parametrically static behavior in $S_{\mu\rho}$ systems falls off to zero when the circuit parameter δ tends to zero. This means that the process dynamics in such closed-loop systems can be made as close as desirable to the specified dynamic performance for an S system with $\boldsymbol{A}$ and $\boldsymbol{b}$ of general form.

It should be recognized, however, that the ultimate properties, i.e. attainment of the desired dynamic performance or independence of S system parameters, is inachievable for the examined type of $S_{\mu\rho}$ systems over the class of admissible controls because for $\delta \to 0$ the gain α in the COFB loop grows without bound and the $S_{\mu\rho}$ system (7.185)-(7.187) degenerates to a VS_h system.

In summary, the disclosed class of binary control algorithms with three types of feedback, CFB, COFB, and OFB, is capable of ensuring the required weak dependence on variable parameters. For constant or jumpwise-variant parameters, parametrical static error is achievable whose magnitude is smaller than any specified number, however small. In some situations this can signify an approach of feedback system properties to those desired, in others a weak dependence on S system parameters. Asymptotic independence of parameters astatic behavior cannot be warranted in all cases, hence a complete solution of the control problem as formulated in Chapter 6 is again inachievable. Therefore in the ensuing chapter we turn to a fourth type of feedback, operator coordinate feedback, and the respective closed-loop control systems.

8 Operator-Coordinate Feedback

The object of our investigation in this chapter will be binary control systems with various COFB laws of Chapters 6 and 7 supplemented with an operator-coordinate feedback loop (OCFB). We shall aim at evaluating the capabilities supplied to the system by OCFB on the way to complete solution of the control problem for parametrically uncertain $\mathcal{P}_x$-processes or S systems.

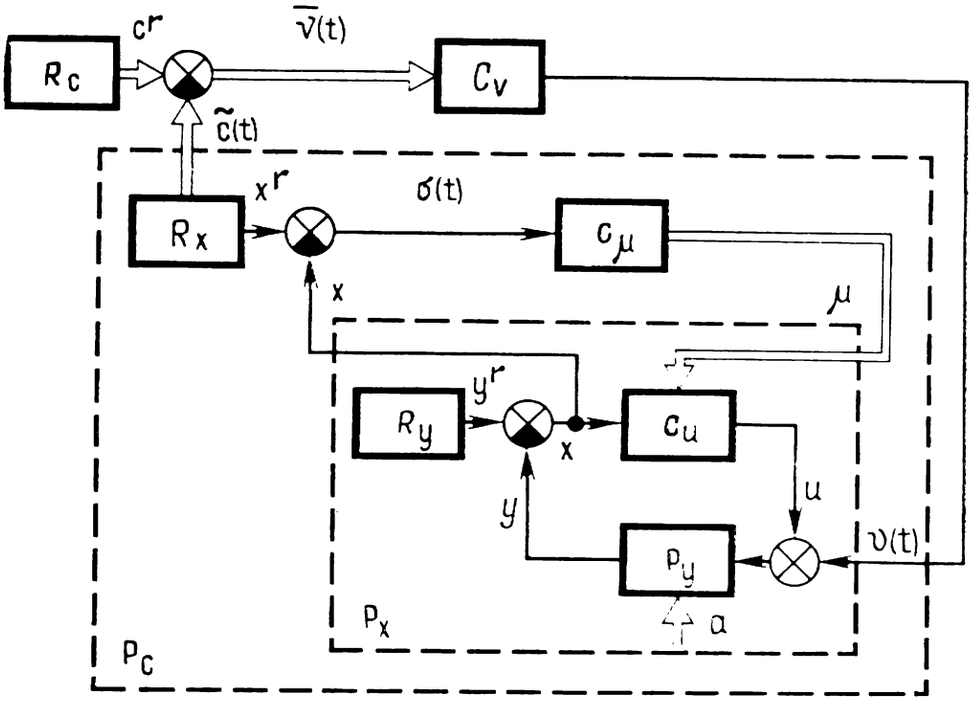

Figure 8.1

The foregoing development has revealed that in binary control systems without OCFB no complete solution is achievable subject to confined controller gains. There have always been controlled processes in such systems whose dynamics may considerably depend on parameters of the controlled process or system. For other controlled processes it can be ensured that either their dynamics will become close to the desired performance specifications, or in some specific cases their dynamics will depend on plant parameters only insignificantly. For these latter cases, one may hope for a weak dependence on parameters or even parametrical static error, but the system can never be made asymptotically independent of parameters as is required for complete solution of the control problem. Incorporation of operator-coordinate feedback is aimed at achieving an asymptotically independent response for all controlled processes rather than for a portion of them.

The block diagrams of binary control systems with OCFB to be examined in this chapter are depicted in Figs. 8.1 and 8.2. Figure 8.1 represents a first type of binary control systems which results from augmenting S_μ systems of Chapter 6 with an operator-coordinate feedback loop, reflected by the feedback operator C_v. This operator represents an OCFB law devised to generate a control $v\,(t)$ from an error signal $\nu\,(t)$ in this feedback loop. This error signal characterizes the deviation of parameters $\tilde{c}\,(t)$ of the dynamic reference input element R^x from the reference law $c^r\,(t)$ specified by the reference input element R_c. Accordingly, $\nu\,(t)$ is an operator type variable, this being graphically reflected by double arrows in

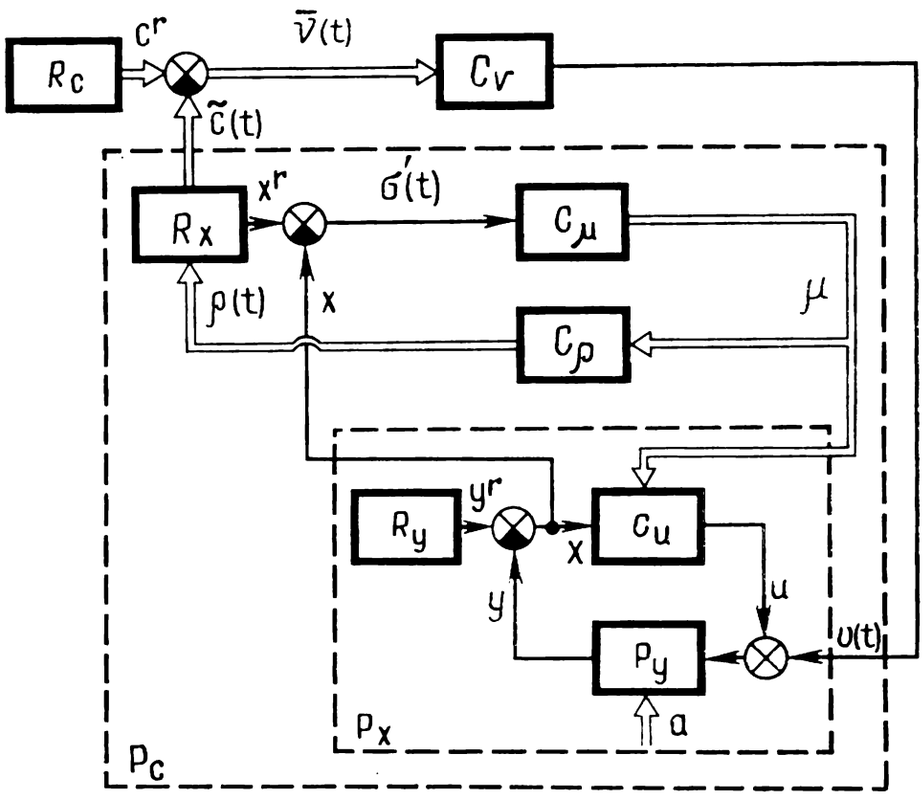

Figure 8.2

Figs. 8.1 and 8.2. The output signal of this loop $v\ (t)$ is added to the output signal $u\ (t)$ of the CFB loop to produce the total control

$$\bar{u}\ (t) = u\ (t) + v\ (t) \tag{8.1}$$

Because $v\ (t)$ is an additive control — in contrast to, say, the multiplicative signal $u\ (t)$ — it naturally relates to the class of coordinate actions. Therefore, operator C_v transforms the operator variable $\nu\ (t)$ into the coordinate variable $v\ (t)$; this was the reason behind calling this loop the operator-coordinate feedback.

Inspection of Fig. 8.2 readily indicates that in addition to the structure of Fig. 8.1 it contains an OFB loop with operator C_ρ. It reflects a type of binary control system resulted from incorporating an operator-coordinate feedback into the $S_{\mu\rho}$ system of Chapter 7.

In agreement with the notation introduced in Chapter 4 we shall refer to control systems generalized in Fig. 8.1 as $S_{\mu\nu}$ systems, whereas those in Fig. 8.2 as $S_{\mu\rho\nu}$ systems. The respective control algorithm will be called $\mathscr{A}_{\mu\nu}$ algorithms and $\mathscr{A}_{\mu\rho\nu}$ algorithms. Schematically, these are represented by the portions depicted beyond the dashed box P_x. The box P_c in these figures represents the generalized plant with the vector output variable $\tilde{\boldsymbol{c}}\ (t)$, scalar control $v\ (t)$, and the vector external disturbance $\boldsymbol{a}\ (t)$ which is arbitrarily variable and separately bounded for every component.

This generalized plant P_c is feedback controlled (feedback operator P_v) with no information available on exterior disturbances

$\boldsymbol{a}(t)$, the control using only some prior bounds of the disturbance. With this system conception, the incorporation of OCFB is aimed at achieving an astatic behavior with respect to the controlled variable $c(t)$, i.e. ensuring equality in the limit, viz. $\lim_{t\to\infty} \| c^r(t) - \tilde{c}(t) \| = \lim_{t\to\infty} \| \nu(t) \| = 0$. For linear control systems such a goal would be unattainable. At best, one could hope for building a dissipative control system, i.e. such a problem statement has no sense for linear systems. For nonlinear systems, on the other hand, such problems are meaningful and can have a variety of solutions.

To complete this introductory section we note that as before the figure following an algorithm symbol in parentheses as q in $\mathscr{A}_{\mu\nu}(q)$ or $\mathscr{A}_{\mu\rho\nu}(q)$ denotes that q components of the state vector $\boldsymbol{x}(t)$ are explicitly involved in the CFB law. The case of $q = 1$ means that the first component is in use.

8.1 OCFB Generation Concepts

In order to construct a binary control system with operator-coordinate feedback along the conceptual lines depicted in Figs. 8.1 or 8.2 we need to define the rules of design for the reference input element R_c which forms $c^r(t)$ and the methods of selecting vector $\boldsymbol{c}(t)$ which is the controlled variable of the plant P_c. Having tackled these problems enables determining the error of the OCFB loop $\nu(t)$. The next to handle is the problem of choosing operator C_v. We shall be concerned here with two types of C_v, one realizing static (proportional) control laws, while the other dynamic (integral and inertial) laws.

There exists, of course, a variety of ways to form the error $\nu(t)$, we are going to realize one based on the following argument. Consider for definiteness the S_μ (1) system of Section 6.7. In the state space for every V solution $[\boldsymbol{x}(t), \mu(t)]$ of this system for $t \geqslant t_0$ we can construct at every $\boldsymbol{x}(t)$ other than O_x the "tracking" surface

$$\sigma^\lambda(\boldsymbol{x}(t)) = \sigma(\boldsymbol{x}(t)) - \delta\lambda(t) \mid x(t) \mid = 0, \quad t \geqslant t_0 \tag{8.2}$$

where $\sigma(\boldsymbol{x}) = -\boldsymbol{c}\boldsymbol{x} = -(c\dot{x} + x)$.

At every $\boldsymbol{x}(t)$ with $\| \boldsymbol{x}(t) \| > 0$, this surface defines on a V solution the continuous function of time

$$\lambda(t) = \sigma(\boldsymbol{x}(t))/\delta \mid x(t) \mid, \quad t \geqslant t_0 \tag{8.3}$$

This function is bounded, $| \lambda(t) | \leqslant 1$, when $\boldsymbol{x}(t) \in G_\delta$, equals zero for $\sigma(\boldsymbol{x}(t)) = 0$ and can grow arbitrarily large when $\boldsymbol{x}(t) \notin G_\delta$.

Geometrically, $\lambda(t)$ describes for each t the distance in angular units to the domain $G_0 = \{x: \sigma(x) = 0\}$. It seems intuitively appealing therefore to relate the error $\nu(t)$ to this function.

To establish such a relation we write the equation of the surface $\sigma^{\lambda}(t)$ in the form $\sigma^{\lambda}(t) = -\boldsymbol{c}^{\lambda}(t)\, x(t) = 0$, where vector $\boldsymbol{c}^{\lambda}(t)$ is defined as

$$\boldsymbol{c}^{\lambda}(t) = \boldsymbol{c} + \delta\lambda(t)\, \boldsymbol{R}^{x}(t) \tag{8.4}$$

and where $\boldsymbol{R}^{x} = [\operatorname{sgn} x, 0]$.

Recall that the Π_x parameter space is defined as the space of coefficients of the respective scalar differential equation. The vector function $\boldsymbol{c}^{\lambda}(t) = [c_1^{\lambda}(t), c_2^{\lambda}(t)]$ is by definition such that $\boldsymbol{c}^{\lambda}(t) \times x(t) = 0$. Since for the S_{μ} system examined $x = [x, \dot{x}]^{\mathrm{T}}$, this equation reads $c_1^{\lambda}(t)\, \dot{x}(t) + c_2^{\lambda}(t)\, x(t) = 0$. Hence, point $c^{\lambda}(t) \in \Pi_x$ characterizes the coefficients of the differential equation at every time instant. The distance from this point to a desirable point $\boldsymbol{c}$ is treated as the OCFB loop error $\nu(t)$.

This argument leads us to take the function $c^{\lambda}(t)$ given by (8.4) as the reference signal of the element R_c. However, this function is unbounded which prevents its using as a reference variable. A more reasonable choice then seems to be the bounded function

$$\boldsymbol{c}^{r}(t) = \boldsymbol{c} + \delta \operatorname{sat}[\lambda(t)]\, \boldsymbol{R}^{x}(t) \tag{8.5}$$

where sat $(\cdot)$ is the saturation function defined in (6.12).

Now $x(t) \in G_{\delta}$ means $\boldsymbol{c}^{s}(t) = \boldsymbol{c}^{\lambda}(t)$, and $x(t) \notin G_{\delta}$ means $\boldsymbol{c}^{r}(t) = \boldsymbol{c} + \delta\, \boldsymbol{R}^{x}(t) \operatorname{sgn} \lambda(t)$.

Finally, we turn to define vector $\boldsymbol{c}$ in Fig. 8.1. Observing that the parameters of the reference input element R_x are constant for the S_{μ} system it would be natural to choose $\tilde{\boldsymbol{c}}$ also constant. Letting $\tilde{\boldsymbol{c}} = \boldsymbol{c} = [1, c]$ we obtain for the OCFB loop error

$$\boldsymbol{\nu}(t) = \delta \operatorname{sat}[\lambda(t)]\, \boldsymbol{R}^{x}(t) \tag{8.6}$$

or observing that the second component of vector $\boldsymbol{\nu}(t)$ is zero in the considered case, the scalar function

$$\nu(t) = \delta \operatorname{sat}[\lambda(t)] \operatorname{sgn} x(t) \tag{8.7}$$

This expression for the OCFB loop error is fairly convenient in $S_{\mu\nu}$ system analysis, therefore we shall exploit it in our further discussion for this purpose. However, practical implementation of $S_{\mu\nu}$ systems with $\nu(t)$ generated in this way is rather difficult. This stems from the fact that $\lambda(t)$ can take arbitrarily large values. We point therefore to another method of generating $\nu(t)$, which gives the same values as (8.7) but relies on bounded functions only.

We observe the fact that $\sigma(t)$ and $\lambda(t)$ are of the same sign. It is not hard to see then that the function

$$\begin{aligned} \nu(t) &= \delta\lambda(t)\,\operatorname{sgn} x(t) \quad \text{for } |\sigma(t)| \leqslant \delta|x(t)| \\ &= \delta\,\operatorname{sgn}[\sigma(t)\,x(t)] \quad \text{for } |\sigma(t)| > \delta\,|x(t)| \end{aligned} \tag{8.8}$$

coincides with the function of (8.7). A similar method of building the reference input element R_c will be in use in constructing $S_{\mu\rho\nu}$ systems, where, however, the equality $\sigma'(t) = \sigma(t) + \delta\mu(t) \times |x(t)| = 0$ may hold under certain conditions for all t beginning from a time $t_1 \geqslant t_0$. Comparing this equality with (8.2) we see that $\lambda(t) = -\mu(t)$, therefore from t_1 the error $\nu(t)$ in $S_{\mu\rho\nu}$ systems takes the form

$$\nu(t) = -\delta\mu(t)\,\operatorname{sgn} x(t), \quad t \geqslant t_1 \tag{8.9}$$

It should be emphasized that whenever $\sigma'(t) = 0$ fails, $\nu(t)$ is defined by (8.7) in $S_{\mu\rho\nu}$ systems as well.

A similar argument may be used for deriving the error in OCFB loops of $S_{\mu\nu}$ (2) and $S_{\mu\rho\nu}$ (2) systems, though the error will always have vector form there.

Let us consider now what control law in OCFB loops would be reasonable to examine in order to achieve a complete solution of the control problem. In the block diagram of Fig. 8.1, this corresponds to selecting a suitable type of operator C_v. First of all we note that the OCFB loop error $\nu(t)$ changes sign along with $x(t)$. This means that for OCFB laws with constant parameters the feedback signal in this loop will be sign variable and the ultimate goal of this feedback ($\nu(t) \to 0$ as $t \to \infty$) cannot be achieved in such setting. Consequently, OCFB is bounded to change sign, as COFB and OFB do, i.e. sign variability is a principal feature of all the new types of feedback introduced in Chapter 4.

In this chapter, as before, we shall focus on examining static (bang-bang and proportional) and dynamic (integral and inertial) OCFB control laws bearing in mind the salient features of this case as found above.

Bang-bang operator-coordinate feedback

For a bang-bang OCFB law, the output variable, i.e. the manipulated variable issued by this control loop, is defined as

$$v(t) = -\gamma\,\operatorname{sgn}[\nu(t)\,x(t)] \tag{8.10}$$

where γ is a constant. This expression rewrites in view of (8.7) and the fact that $\operatorname{sgn}\lambda = \operatorname{sgn}\sigma$ as

$$v(t) = -\gamma\,\operatorname{sgn}\sigma(t) \tag{8.11}$$

This expression defines the discontinuous component of the control $u(t)$ defined in (8.1). This component is discontinuous on the

domain $G_0 = \{x: \sigma(x) = 0\}$ which is exactly the domain to be made invariant for the controlled processes in the binary systems under examination. Therefore, if with the OCFB control law (8.10) the control problem is completely solvable, the solution will be achieved with a sliding mode occurring on G_0. In this case $u(t)$ no longer belongs to the class of continuous functions of time required in the control problem formulation, and the feedback law $u[x, t]$ itself is outside the feasible class [see (6.1)] of feedback laws. Consequently, under the stated conditions the bang-bang OCFB law does not suit solution of the control problem in much the same way as the bang-bang COFB law.

Note for comparison that additional control $v(t)$ such as (8.11) has been used earlier, for example, in variable structure systems. It has been incorporated for formal reasons without due meaningful reasoning. In the conceptual framework of the new types of feedback presented in this book, adding bang-bang actions can be interpreted in standard terms of automatic control. They represent a particular type (bang-bang) of control laws in OCFB loops devised to fulfill accurately or asymptotically the specifications on system's dynamic performance.

In terms employed in this book, variable structure systems relate to $S_{\mu\nu}$ (respectively $S_{\mu\rho\nu}$) systems with bang-bang control in COFB and OCFB loops. When other than a relay law is used in the COFB loop, then such control systems are no longer VS systems even with a bang-bang OCFB law, rather they are binary systems governed by the type of control in COFB and OFB loops. Binary systems with bang-bang OCFB are interesting objects for a separate study. Indeed, every component of control in VS systems with bang-bang action is discontinuous, which, as has been already noted above, may present certain difficulties for system's actuators.

In binary control systems with bang-bang OCFB, control is also discontinuous, but with a sufficiently small parameter γ in the law (8.11) the magnitudes of these discontinuities will be insignificant while the main component of control will be continuous. This method of organizing sliding modes on a desired manifold may seem preferable from a number of aspects. Specifically, it may cause more favorable operating conditions for actuators as in this case they do not have to handle reverse commands. We are not going to dwell on binary control systems with bang-bang OCFB any longer as the controls they generate lie outside the class of admissible control laws.

Proportional operator-coordinate feedback

Consider two methods of constructing such feedback. One has the output variable of the loop, $v(t)$, proportional to the error $\nu(t)$ and the coordinate $x(t)$, i.e.

$$v(t) = -\gamma\nu(t)\, x(t) \tag{8.12}$$

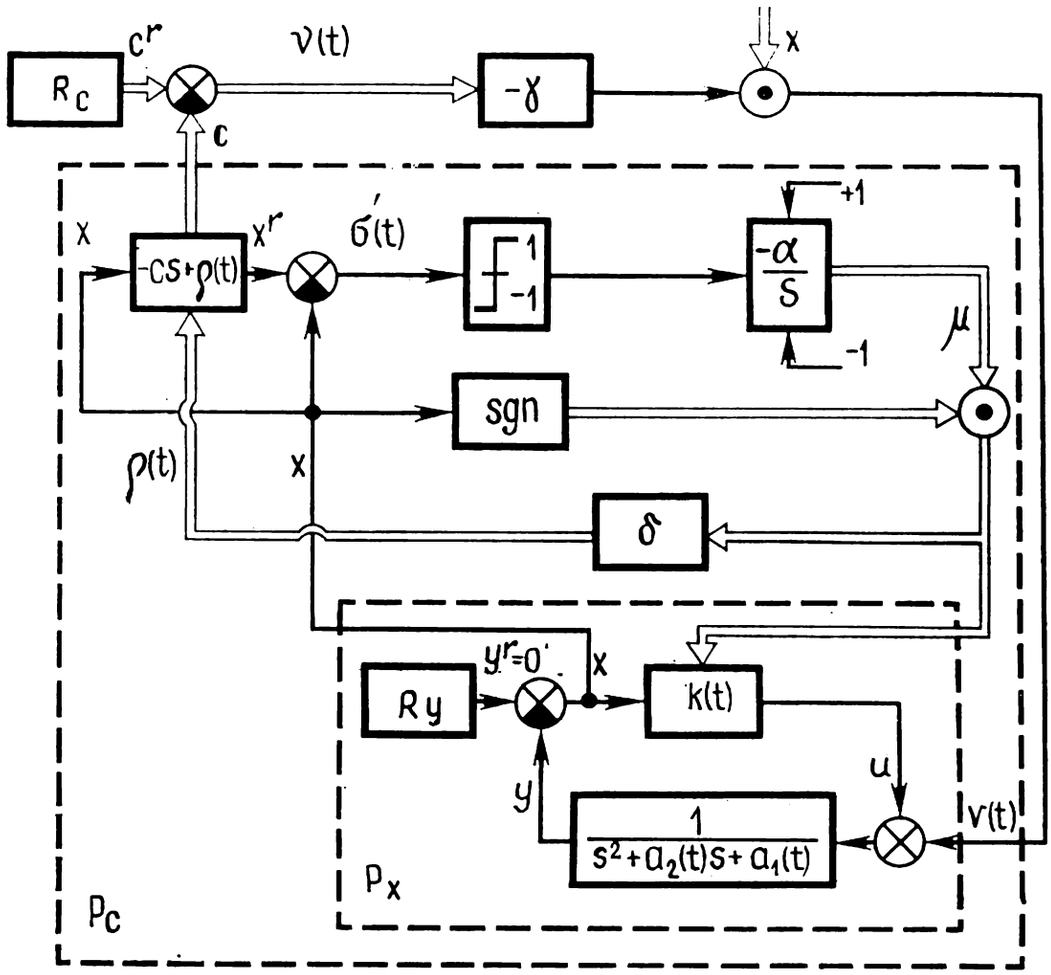

Figure 8.3

or upon substitution of $\nu\ (t)$ from (8.7)

$$v\ (t) = -\gamma\delta\ |\ x\ (t)\ |\ \text{sat}\ (\lambda\ (t)) \tag{8.13}$$

This method of OCFB conceptualization transforms the block diagram of Fig. 7.3 into the block diagram of the $S_{\mu\rho\nu}$ system presented in Fig. 8.3. This figure does not disclose the structure of the reference input elements R_c. It is assumed that this operator realizes the function following from equations (8.5)-(8.8), namely,

$$\begin{aligned} \boldsymbol{c}^r\ (t) &= \boldsymbol{c} + \delta\lambda\ (t)\ \text{sgn}\ x\ (t) \quad \text{for} \quad |\ \sigma\ (t)\ | \leqslant \delta\ |\ x\ (t)\ | \\ &= \boldsymbol{c} + \delta\ \text{sgn}\ [\sigma\ (t)\ x\ (t)] \quad \text{for} \quad |\ \sigma\ (t)\ | > \delta\ |\ x\ (t)\ | \end{aligned}$$

This representation is connected with the fact that there is no need for a reference input element R_c in a practical realization of such an $S_{\mu\rho\nu}$ system. In this case we need to exclude all internal variable to simplify the expression for the error $\nu\ (t)$ and to implement thus derived relationship.

The incorporation of the element R_c into the system is worthwhile at the stage of structural synthesis of the control system when conceptual considerations are overriding.

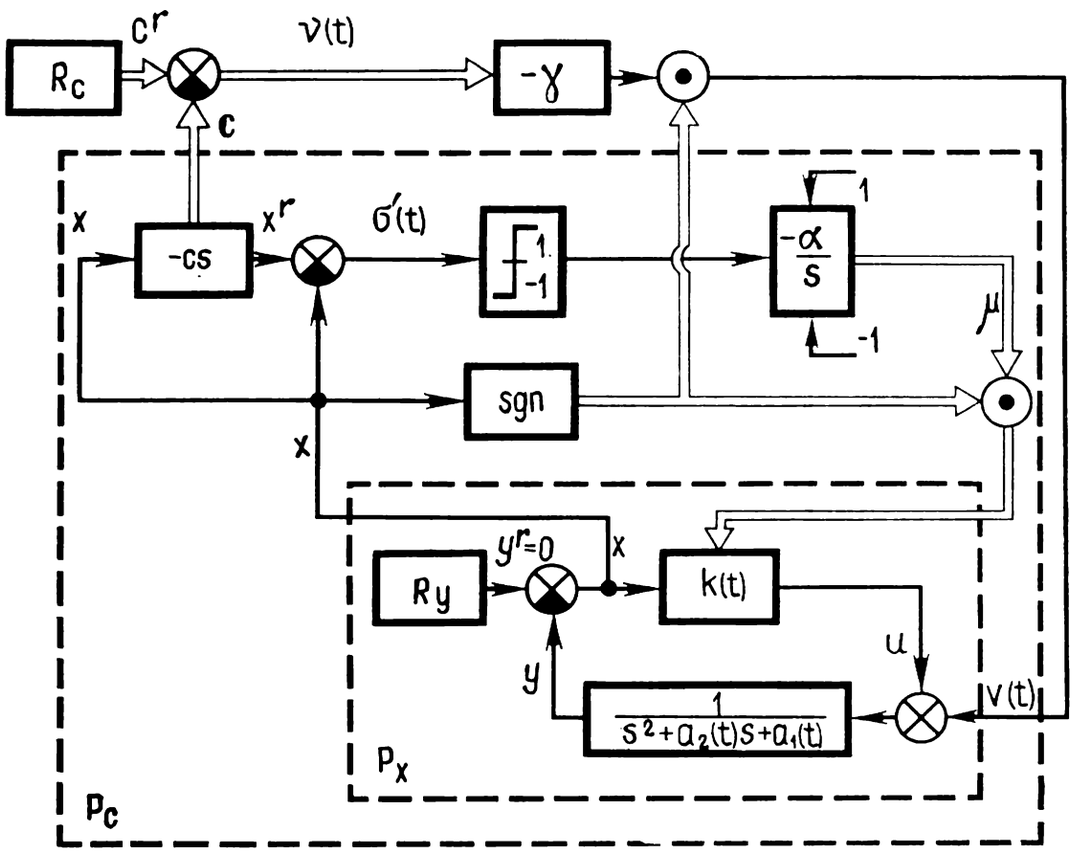

Figure 8.4

Let us now examine whether such OCFB conceptualization is helpful in achieving our control objective. For this purpose it will suffice to establish the values of control, defined by 8.1, at various stages of motion in the $S_{\mu\rho\nu}$ system. Indeed, if the controlled process $x(t)$ does not belong to $G_\delta = \{x: \mid \sigma(x) \mid \leqslant \delta \mid x \mid\}$ on a time interval $2/\alpha$ long, then, as can be immediately seen from (8.13), (8.1), (7.16), and (7.17), the control $u(t)$ takes the form

$$u(t) = -(k^0 + \gamma\delta) \mid x(t) \mid \operatorname{sgn} \sigma(t) \tag{8.14}$$

If now $x(t) \in G_\delta$ and $\sigma'(t) = \sigma(t) + \delta\mu(t) \mid x(t) \mid = 0$, then, as can be seen from the foregoing development, $\lambda(t) = -\mu(t)$ so that $v(t) = \gamma\delta \mid x(t) \mid \mu(t)$. Substituting this expression into (8.1) leads to

$$u(t) = (k^0 + \gamma\delta)\,\mu(t) \mid x(t) \mid \tag{8.15}$$

Comparison of equations (8.14) and (8.15) with the respective equations for control in the $S_{\mu\rho}$ system (7.16) at $\mu = -\operatorname{sgn}\sigma$ indicates that the OCFB (8.13) causes a change in CFB loop gains which means that the properties of the $S_{\mu\rho\nu}$ system, corresponding to the block diagram of Fig. 8.3, are equivalent to those of $S_{\mu\rho}$ systems of Section 7.1. Accordingly, with the findings of Section 7.1 we are in a position to state that the use of OCFB such as (8.13) cannot secure complete solution of the control problem, i.e. again we cannot answer this question in affirmative.

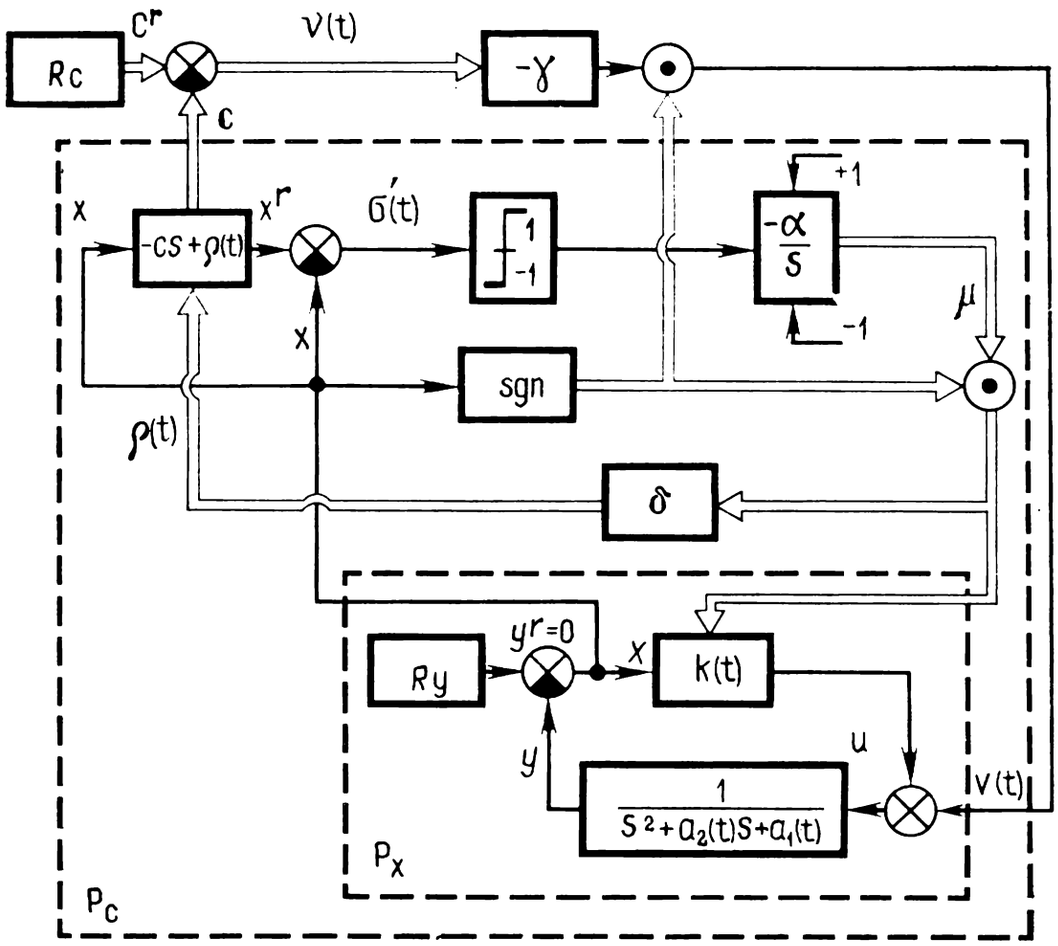

Figure 8.5

The second method of building proportional operator-coordinate feedback resorts to the following control law:

$$v\,(t) = -\gamma\nu\,(t)\ \mathrm{sgn}\ x\,(t) \tag{8.16}$$

which after incorporation of (8.7) rewrites as

$$v\,(t) = -\gamma\delta\ \mathrm{sat}\ (\lambda\,(t)) \tag{8.17}$$

Figures 8.4 and 8.5 present the block diagrams of $S_{\mu\nu}$ and $S_{\mu\rho\nu}$ systems, respectively, which occur by augmenting the S_{μ} system of Section 6.7 and the $S_{\mu\rho}$ system of Section 7.1 by OFCB as outlined above. Note that the sign of $x\,(t)$ has been introduced into the OCFB law (8.16) to make this feedback sign-constant. Below we shall give such systems a thorough consideration. Brief comments will be given also on other binary control systems with other COFB and OCFB laws. Specifically binary control systems with dynamic OCFB laws will be examined.

Dynamic operator-coordinate feedback laws

It will be recalled that the objective of OCFB is making the error $\nu\,(t)$ vanish in the presence of unbounded external disturbances $a\,(t)$. In this respect, the use of dynamic rather than static

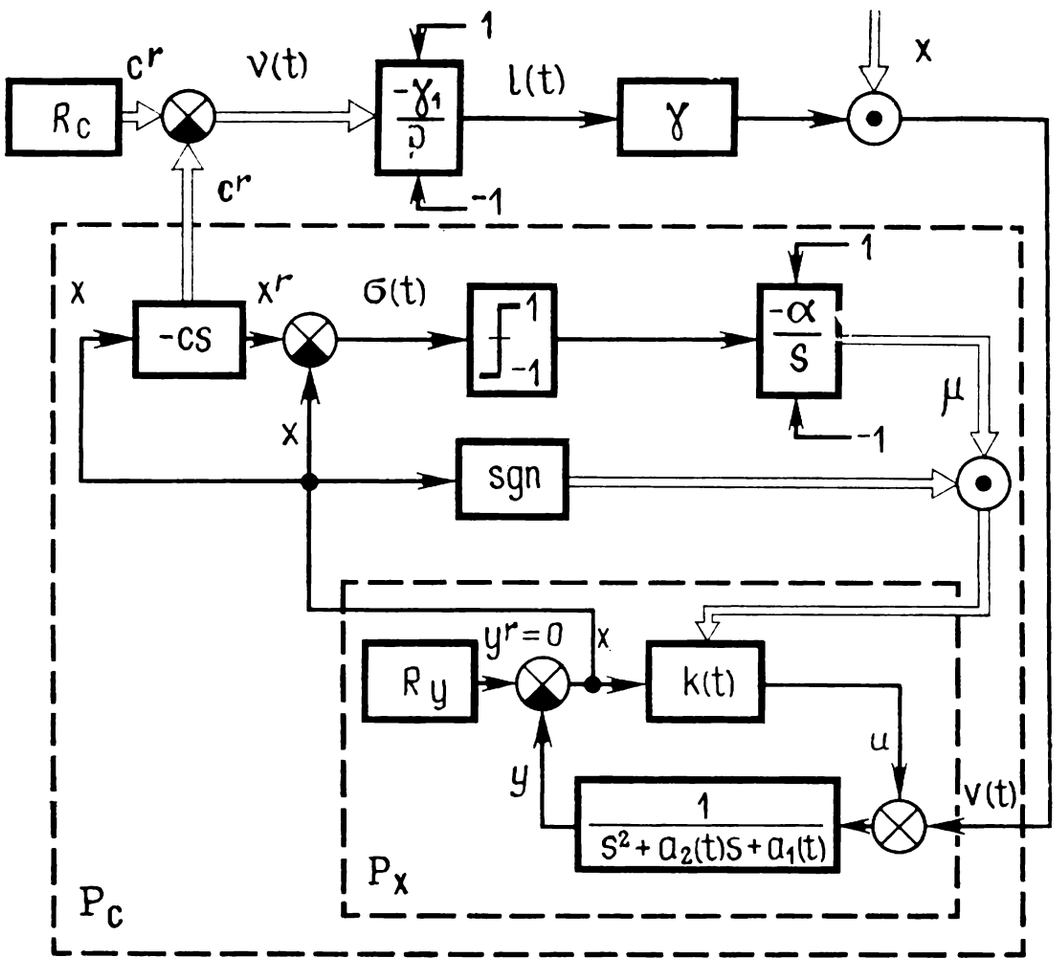

Figure 8.6

control laws in the OCFB loop is a more sound proposition as suggested by traditional techniques of control theory. In particular, with stepwise variation of $\mathcal{P}_x$ parameters, the problem may hopefully be solved with resort to an integral control law.

Two versions of integral OCFB design are possible. By one method we assume that the OCFB loop error is formulated as above and consider the $S_{\mu\nu}$ and $S_{\mu\rho\nu}$ systems which result from supplementing S_μ and $S_{\mu\rho}$ systems with integral COFB by integral OCFB of the form

$$\begin{aligned} \nu(t) &= \gamma x(t)\, l(t) \\ \dot{l}(t) &= -\gamma_1 \nu(t) \quad \text{for } |l(t)| \leqslant 1 \\ &= -\omega_1 l(t) \quad \text{for } |l(t)| > 1,\ |l(t_0)| \leqslant 1,\quad t \geqslant t_0 \end{aligned} \tag{8.18}$$

where γ and γ_1 are some constants augmenting the algorithm parameter set, ω_1 a positive constant, and $l(t)$ a new variable constrained with $|l(t)| \leqslant 1$ for $t \geqslant t_0$.

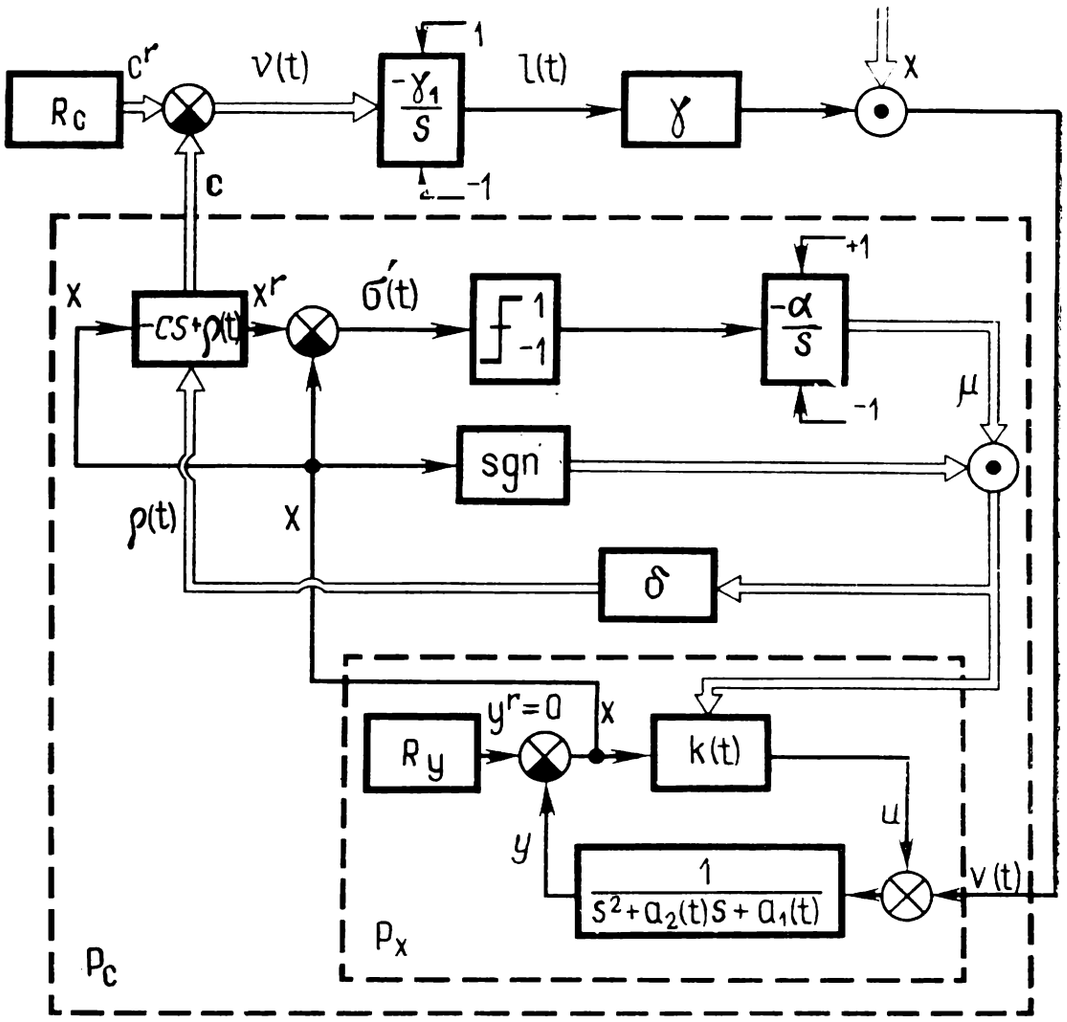

Figure 8.7

Substituting (8.7) into (8.8) produces a COFB law in the form

$$v(t) = \gamma x(t)\, l(t)$$

$$\begin{aligned} \dot{l}(t) &= -\gamma_1 \operatorname{sat}(\lambda(t)) \operatorname{sgn} x(t) && \text{for } |l(t)| \leqslant 1 \\ &= -\omega_1 l(t) && \text{for } |l(t)| > 1,\ |l(t_0)| \leqslant 1,\quad t \geqslant t_0 \end{aligned} \tag{8.19}$$

The block diagrams of the $S_{\mu\nu}$ and $S_{\mu\rho\nu}$ systems equipped with this OCFB law are depicted in Figs. 8.6 and 8.7.

The second version of building integral OCFB differs from the above in the way of accounting for the sign-variable nature of error $\nu(t)$. Whereas in the former method the constant sign of OCFB was ensured by multiplying the function $\gamma l(t)$ by $x(t)$, now this property is fixed by multiplying $\nu(t)$ by the sign of $x(t)$. The associated OCFB control laws have the form

$$v(t) = \gamma |x(t)|\, l(t)$$

$$\begin{aligned} \dot{l}(t) &= -\gamma_1 \nu(t) \operatorname{sgn} x(t) && \text{for } |l(t)| \leqslant 1 \\ &= -\omega_1 l(t) && \text{for } |l(t)| > 1,\ |l(t_0)| \leqslant 1,\quad t \geqslant t_0 \end{aligned} \tag{8.20}$$

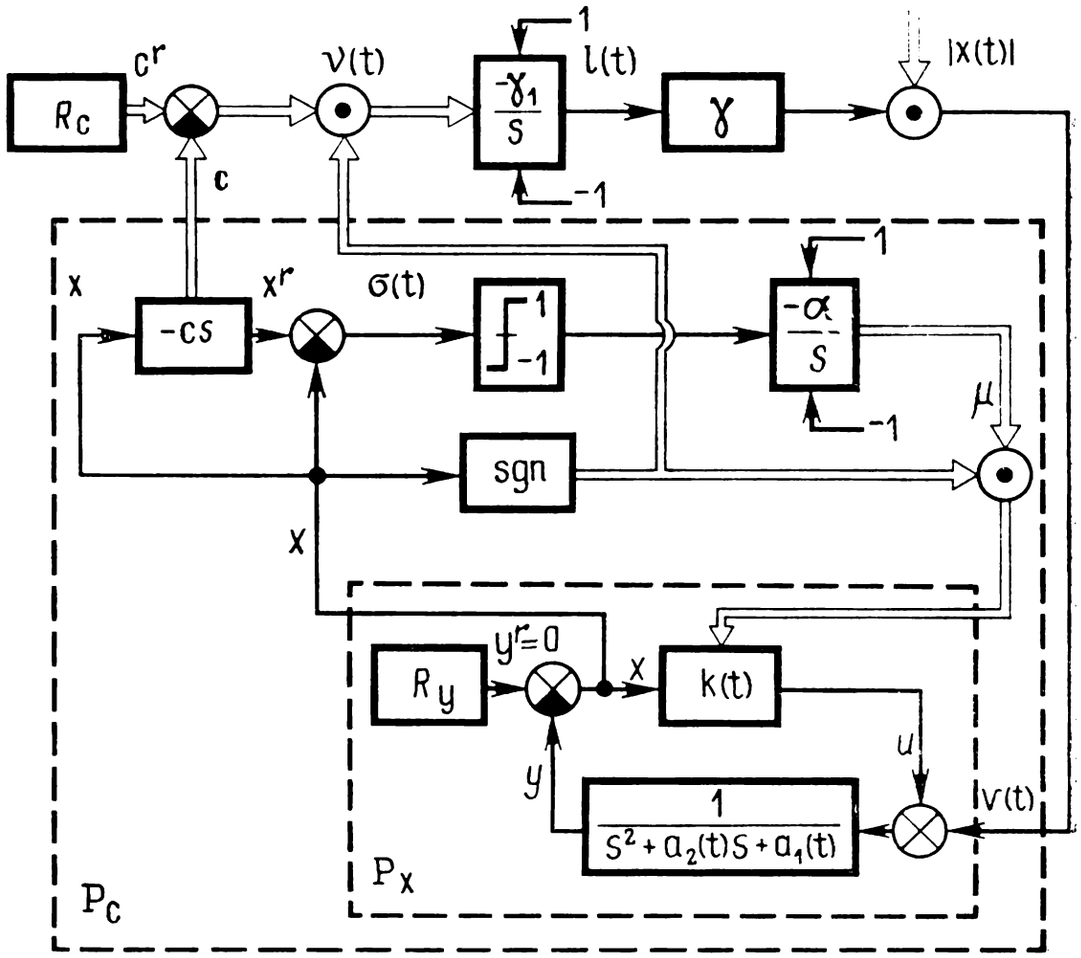

Figure 8.8

or after incorporating $\nu(t)$ from (8.7)

$$v(t) = \gamma |x(t)| l(t)$$

$$\begin{aligned} \dot{l}(t) &= -\gamma_1 \operatorname{sat}(\lambda(t)) \quad \text{for } |l(t)| \leqslant 1 \\ &= -\omega_1 l(t) \quad \text{for } |l(t)| > 1, \ |l(t_0)| \leqslant 1, \ t \geqslant t_0 \end{aligned} \tag{8.21}$$

The block diagrams for the $S_{\mu\nu}$ (1) and $S_{\mu\rho\nu}$ (1) systems with integral COFB incorporating the above OCFB are presented in Figs. 8.8 and 8.9, respectively.

The OCFB laws (8.9) and (8.11) differ though insignificantly in the following. It is straightforward to show that if $x(t)$ retains its sign on a time interval, then signals $v(t)$ formed in accord with these laws coincide. In particular, $x(t)$ remains sign-constant when $x(t) \in G_\delta$. Consequently, the properties of processes in these systems coincide once they belong to G_δ. A distinct behavior of $x(t)$ takes place outside G_δ. It appears when the controlled process crosses the axis $x = 0$. Under the law (8.19) $x(t)$ changes sign at this instant and the output quantity of the integral $l(t)$ in Figs. 8.5 and 8.6 reverses sign within the time interval $2/\gamma_1$ long. This effect does not occur in control systems with the law (8.21). Besides, this method

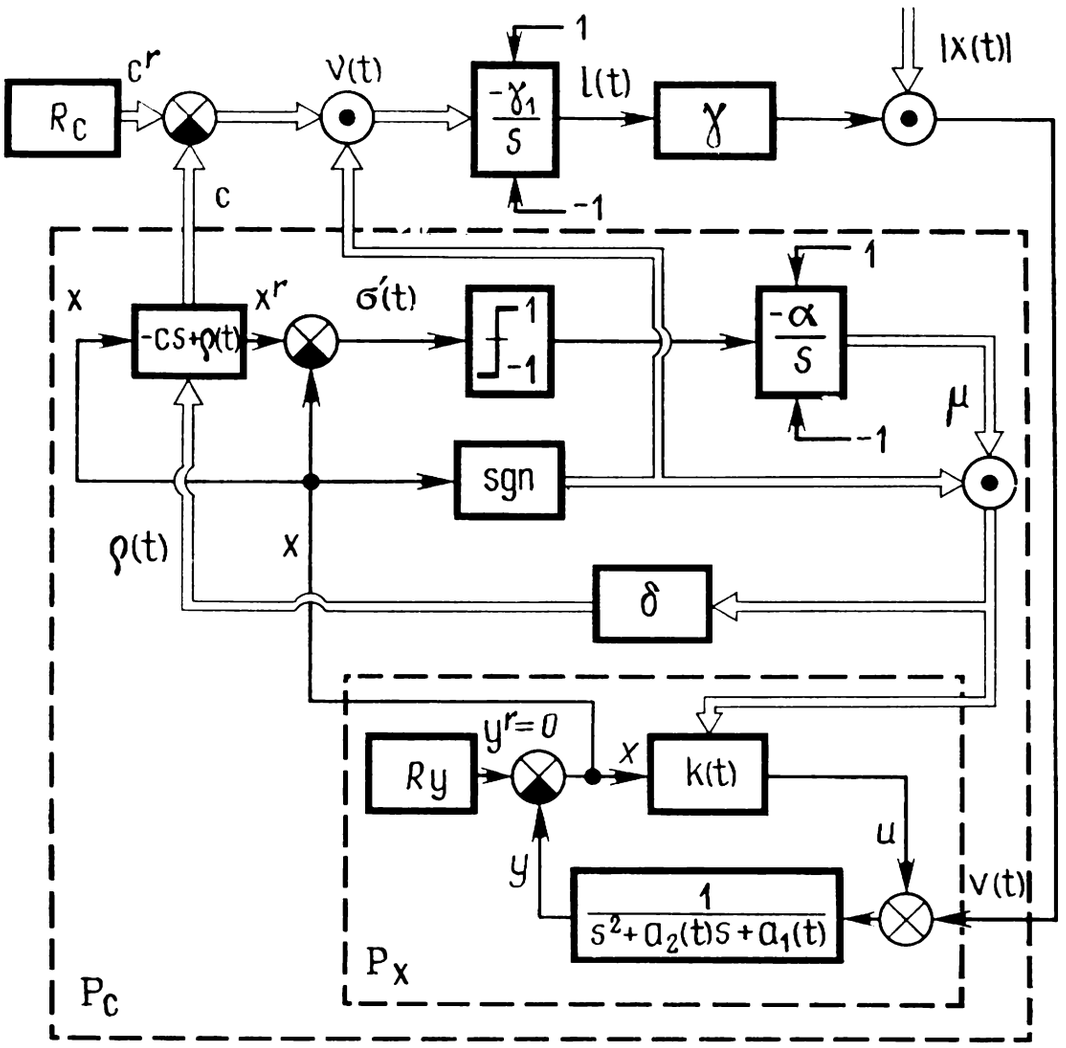

Figure 8.9

of integral OCFB design is more convenient for analytical studies. Therefore in what follows we shall stick to this type of control systems.

Finally, some attention will be given to binary control systems with inertial OCFB and COFB laws. In this direction, only $S_{\mu\rho\nu}$ systems will be considered with OCFB laws such as

$$\begin{aligned} v\,(t) &= \gamma l\,(t)\mid x\,(t)\mid \\ \dot{l}\,(t) &= -\gamma_1\,[l\,(t) + \operatorname{sgn}\,(\nu\,(t)\,x\,(t))], \quad t \geqslant t_0 \end{aligned} \tag{8.22}$$

where γ and γ_1 are constant circuit parameters of the control algorithm. Substituting the equation for $\nu\,(t)$ carries this law to

$$\begin{aligned} v\,(t) &= \gamma l\,(t)\mid x\,(t)\mid \\ \dot{l}\,(t) &= -\gamma_1\,[l\,(t) + \operatorname{sgn}\,\sigma\,(t)], \quad t \geqslant t_0 \end{aligned}$$

With this OCFB law, the block diagram of $S_{\mu\rho}$ system with inertial COFB looks like that in Fig. 8.10.

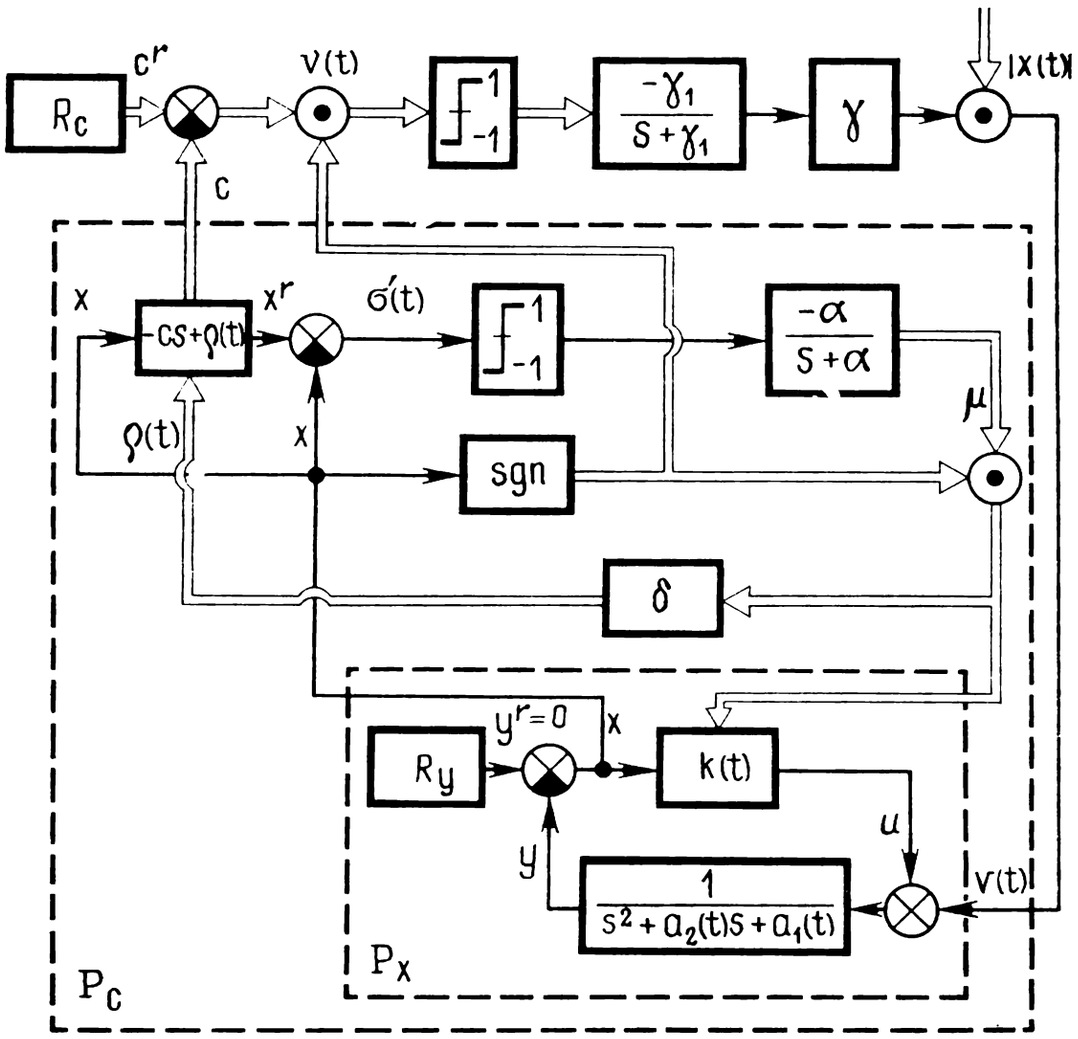

Figure 8.10

One remark is in order with respect to the relation of the total control $\boldsymbol{u}$ (8.21), if employed in binary systems with the types of OCFB outlined above, to the class of admissible controls (6.1). For control systems with integral and inertial OCFB, equations (8.20), (8.22) and the CFB loop control $u\ (\boldsymbol{x},\ t) = \mu\ (t)\ k^0 \mid x \mid$ yield the total control in the form $\boldsymbol{u}\ (\boldsymbol{x},\ t) = [\mu\ (t)\ k^0 + l\ (t)\ \gamma] \mid x \mid$ which, clearly, satisfies the conditions (6.1). For the static OCFB (8.17), the total control has the form $\boldsymbol{u}\ (\boldsymbol{x},\ t) = \mu\ (t)\ k^0 \mid x \mid - \gamma\delta\ \mathrm{sat}\ \lambda\ (\boldsymbol{x})$. This control law, in general, does not belong to the admissible class. Nevertheless we take up this type of control system into consideration because, first, the associated control $\boldsymbol{u}\ (t)$ in these systems is continuous and, second, this method of OCFB organization ensures for all controlled processes a weak (vanishing as $t \to \infty$) dependence of their dynamics on $\mathcal{P}_x$ parameters.

8.2 $S_{\mu\nu}$ (1) Systems with Proportional and Integral Coordinate-Operator Feedback

Consider an $S_{\mu\nu}$ (1) system with the block diagram shown in Fig. 8.4. The associated $\mathscr{A}_{\mu\nu}$ (1) control algorithm is given by the following set:

$$
\begin{aligned}
&\boldsymbol{u}(t) = u(t) + v(t)\\
&u(t) = \mu(t)\, k^0\, |x(t)|\\
&v(t) = -\gamma\delta \text{ sat } \lambda(t)\\
&\dot{\mu}(t) = -\alpha \operatorname{sgn} \sigma(t) \quad \text{for } |\mu(t)| \leqslant 1\\
&\qquad = -\omega\mu(t) \qquad \text{for } |\mu(t)| > 1,\ |\mu(t_0)| \leqslant 1,\quad t \geqslant t_0\\
&\sigma(t) = -c\boldsymbol{x}(t) = -(c\dot{\boldsymbol{x}} + x)\\
&\lambda(t) = \sigma(t)/\delta|x(t)|
\end{aligned}
\tag{8.23}
$$

Taken together with the $\mathscr{P}_x$-process equation

$$
\ddot{x}(t) + a_2(t)\,\dot{\boldsymbol{x}}(t) + a_1(t)\,x(t) = -\boldsymbol{u}(t) \tag{8.24}
$$

where $a_1(t)$ and $a_2(t)$ are subject to (4.2), equations (8.23) define the equations of motion for the $S_{\mu\nu}$ (1) system under investigation. This is a third order dynamic system whose solutions are pairs of functions $[\boldsymbol{x}(t),\ \mu(t)]$ among which the initial conditions in $\mu(t)$ single out V solutions. Those V solutions for which $x(t_0) = O_x$ will be examined separately.

First of all determine the conditions stearing a controlled process $\boldsymbol{x}(t)$, which is a component of a V solution to the $S_{\mu\nu}$ system (8.13)-(8.24), in a final time into the domain $G_\delta = \{\boldsymbol{x}:\ |\sigma(x)| \leqslant \delta\, |x|\}$.

Problem C. Consider for $t \geqslant t_0$ a V solution $[\boldsymbol{x}(t),\ \mu(t)]$ of the $S_{\mu\nu}$ system (8.23)-(8.24) assuming that for $t \in [t_0,\ t_0 + 2/\alpha]$ $\boldsymbol{x}(t) \notin G_\delta$. Then for $t \geqslant t_1 = t_0 + 2/\alpha$ $\mu(t) = -\operatorname{sgn}\sigma(t)$ and sat $\lambda(t) = \operatorname{sgn}\sigma(t)$. Under these conditions for $t \geqslant t_1$

$$
\dot{\sigma} = -\frac{ca_2 - 1}{c}\sigma - \left(\frac{ca_2 - 1}{c} - ca_1\right)x - (ck^0|x| - c\gamma\delta)\operatorname{sgn}\sigma \tag{8.25}
$$

With a positive γ the sign of the last term is always opposite to that of $\sigma(t)$, therefore given the condition $\inf ca_2(t) > 1$ of (6.67) and $k^0 > \tilde{k}^0$, where $\tilde{k}^0 = \sup\limits_{t \geqslant t_0} |\, a_1(t) - a_2/c + 1/c^2\,|$, the function $\sigma(t)$ is monotonously decreasing with $|\sigma(t)| \leqslant |\sigma(t_1)| - c\gamma\delta \times (t - t_1)$. This implies that there is a finite time t_2 when $\boldsymbol{x}(t) \in \partial G_\delta$.

Let us demonstrate now that under some additional conditions every controlled process $\boldsymbol{x}(t)$ starting at $t = t_2$ on the boundary ∂G_δ reaches in finite time the domain G_0 if $\mu(t_2) = -\operatorname{sgn}\sigma(t_2)$.

We have to prove this fact because in binary control systems without OFB the property of $x(t) \in G_\delta$ is valid for only those processes which start from G_0.

To derive the respective sufficient conditions we write the integral equation describing the variation of $\lambda(t)$ for $t \geqslant t_2$ as follows:

$$\lambda(t) = \lambda(t_2) + \int_{t_2}^{t} \frac{\dot{\sigma} - \delta\lambda|x|^{\bullet}}{\delta|x|}\, d\tau$$

$$= \lambda(t_2) + \frac{c}{\delta}\int_{t_2}^{t} [(a_1 - a_2\zeta + \zeta^2)\operatorname{sgn} x - k^0 \operatorname{sgn}\lambda - (\gamma\delta/|x|)\operatorname{sat}(\lambda)]\, d\tau \tag{8.26}$$

where $\zeta = (1 + \xi)/c$, $\xi(t) = \delta\lambda(t) \operatorname{sgn} x(t)$, and it is allowed for $\operatorname{sgn}\sigma = \operatorname{sgn}\lambda$.

Analysis of the integrand in this equation indicates that given $k^0 > \overline{\overline{k}}^0$, with $\overline{\overline{k}}^0$ defined by (6.72), the function $|\lambda(t)|$ monotonously decreases at a finite rate so that there exists a time t_3 when $\lambda(t_3) = 0$. This identity is equivalent to $\sigma(t_3) = 0$ which implies that $x(t_3) \in G_0$. The above argument leads us to formulate the following statement.

Theorem 8.2-1. Suppose with some positive constants c and δ the conditions (6.67) are met and

$$k^0 > \overline{\overline{k}}^0 \tag{8.27}$$

where the constant $\overline{\overline{k}}^0$ is defined by (6.72), then for any positive constant all controlled processes $x(t)$ in the $S_{\mu\nu}$ system (8.23)-(8.24) reach the domain G_0 in a finite time.

Comparing this result with similar findings for binary control systems without OCFB we note that in $S_{\mu\nu}$ systems with static OCFB, x^{II} processes (i.e. those which asymptotically tend to zero as $t \to \infty$ so that $x^{II} \notin G_\delta$) no longer exist and there remain only x^{I} processes. This result means that all controlled processes can be provided with the desired dynamics and dependence on plant parameters. Indeed, once a process appears in G_0 and for $t \geqslant t_3$ remains in G_δ, the bound (6.58) holds, viz. $\|x(t)\| \leqslant [(1 + c + \delta)/c] \times \|x(t_3)\| \exp[-(1 - \delta)(t - t_3)/c]$, so that for $\delta \in [0, 1]$ all controlled processes of the $S_{\mu\nu}$ system converge to the origin of the state space. In addition, for a sufficiently small δ the dynamics of x's is close to the specified and their dependence on plant parameters is insignificant.

Let us establish relationships ensuring $x(t) \in G_\delta$ for $t \geqslant t_0$ if $x(t_0) \in G_0$.

Problem B. This problem is solved by the following theorem.

Theorem 8.2-2. Suppose with some positive constants c and δ, $0 < \delta < 1$ the condition (6.67) is met and

$$k^0 > \max(\tilde{k}^0, \bar{k}^0) \tag{8.28}$$

$$\alpha \geqslant (2c/\delta)(k^0 + \bar{\bar{k}}^0) \tag{8.29}$$

where the constants $\tilde{k}^0$, $\bar{k}^0$, and $\bar{\bar{k}}^0$ are defined by (6.94), (6.86), and (6.72). Then for any $\gamma > 0$ the domain G_δ is G_0 invariant for the $S_{\mu\nu}$ system (8.23)-(8.24).

Prior to proving this theorem we wish to formulate an auxiliary statement disclosing some peculiarities in the behavior of V solutions to the $S_{\mu\nu}$ system (8.23)-(8.24).

Lemma 8.2-1. Under the conditions of Theorem 8.2-2 every V solution $(x(t), \mu(t))$ of the $S_{\mu\nu}$ system (8.23)-(8.24) such that $(x(t_0) = O_x$, for all $t > t_0$ has the form $(O_x, \mu(t))$ with $|\mu(t)| \leqslant 1$.

Proof. Consider for $t \geqslant t_0$ a V solution of the $S_{\mu\nu}$ system examined assuming that $x(t_0) = O_x$ and $|\mu(t_0)| \leqslant 1$. For $\|x\| = 0$ the function $\lambda(x) = \sigma(x)/\delta |x|$ is indefinite and hence so is the control $v = -\gamma\delta \operatorname{sat} \lambda$. Therefore we have to prove that the V solution in this case has the form $(O_x, \mu(t))$. Let us assume the contrary, i.e. for t posterior and close to t_0 $\|x(t)\| > 0$. Consider two cases: $|\sigma(t)| > \delta |x(t)|$ for the said t, and $|\sigma(t)| \leqslant \delta |x(t)|$ for the same t.

Case 1. Assume for definiteness that in this case $\sigma(t) > 0$ (the line of argument for $\sigma(t) < 0$ is similar). Let us write the equation for the variation of $\sigma(t)$ on the V solution under investigation. With account of

$$\sigma(t_0) = 0, \quad \operatorname{sat}\lambda(t) = \operatorname{sgn}\sigma(t) = 1, \quad \text{and } \dot{x} = -(\sigma + x)/c \quad \text{we get}$$

$$\begin{aligned}\sigma(t) &= \int_{t_0}^{t} \frac{d\sigma(t)}{dt}\, d\tau \\ &= c\int_{t_0}^{t}\left[-\frac{ca_2 - 1}{c^2}\sigma + \left(a_1 - \frac{a_2}{c} + \frac{1}{c^2}\right)x + \mu k^0 |x| - \gamma\delta\right] d\tau\end{aligned} \tag{8.30}$$

Estimating this integral with account of (8.28) and (8.29) yields

$$\begin{aligned}\sigma(t) &\leqslant -\int_{t_0}^{t}\frac{ca_2 - 1}{c}\sigma\, d\tau + c\int_{t_0}^{t}[(\tilde{k}^0 + k^0)|x| - \gamma\delta]\, d\tau \\ &\leqslant -\int_{t_0}^{t}\frac{ca_2 - 1}{c}\sigma\, d\tau + c\int_{t_0}^{t}\left(\frac{\alpha\delta}{2}|x| - \gamma\delta\right) d\tau \\ &\leqslant -\int_{t_0}^{t}\frac{ca_2 - 1}{c}\sigma\, d\tau + c\int_{t_0}^{t}\frac{\alpha\sigma - 2\gamma\delta}{2}\, d\tau\end{aligned}$$

For t close to t_0 the first component of the sum is negative in view of (6.67), and the second term is negative owing to the continuity of

$\sigma(t)$, therefore at the considered t $\sigma(t) < 0$ contrary to our assumption. Thus, if a controlled process starts from O_x then $|\sigma| \leqslant \delta |x|$, i.e. the situation corresponds to the second case.

Case 2. Given $|\sigma(t)| \leqslant \delta |x(t)|$ for t close to t_0, by Theorem 6.7-1 the norm of the process satisfies the bound (6.58). Consequently, the considered process remains in an arbitrary neighborhood of O_x, however small, hence $\|x(t)\| = 0$. The lemma is proved.

Comments on the proof of the theorem. The elaborated proof, omitted here, can be readily restored from the proof of Theorem 6.7-2. In our analysis it suffices by virtue of Lemma 8.2-1 to consider a V solution $(x(t), \mu(t))$ such that $|x(t)| > 0$ for $t \geqslant t_0$. Recall that under these conditions and $\lambda(t_0) = 0$ we can write for $\lambda(t)$ an equation similar to (6.63) and unlike it having the term $-\gamma\delta\lambda(t)/|x(t)|$ due to the control $v(t)$, viz.

$$\lambda(t) = \frac{c}{\delta}\int_{t_0}^{t} [(a_1 - a_2\zeta + \zeta^2)\operatorname{sgn} x + k^0\mu - \gamma\delta\lambda/|x|]\,d\tau$$

where $\zeta = (1 + \xi)/c$, and $\xi(t) = \delta\lambda(t)\operatorname{sgn} x(t)$.

Neglecting the last term in the integrand we obtain with the argument of the proof of Theorem 6.7-2 that if at $t = t_1$ the process attains the boundary of the domain G_δ then $\mu(t_1) = -\operatorname{sgn}\sigma(t_1)$ subject to (8.29). Next assume for definiteness that $\sigma(t_1) > 0$ and put down the equation for $\sigma^-(t) = \sigma(t) - \delta|x(t)|$ for $t \geqslant t_1$ with account of $\sigma^-(t_1) = 0$

$$\sigma^-(t) = \int_{t_0}^{t} c\,[(a_1 - a_2\zeta + \zeta^2)\,x - k^0|x| - \gamma\delta\operatorname{sat}\lambda]\,d\tau$$

where $\zeta = (1 + \xi)/c$.

Because for t posterior and close to t_1 $\lambda(t) > 0$, then given (8.28) $\sigma^-(t) < 0$ for t close to t_1. This, however, implies that $|\sigma(t)| \leqslant \delta|x(t)|$ and therefore for all $t \geqslant t_0$ $x(t) \in G_\delta$. Q.E.D.

Thus, under the conditions of Theorems 8.2-1 and 8.2-2 from some time instant t_1 every process $x(t)$ in the $S_{\mu\nu}$ system belongs to G_δ and, consequently, converges to O_x subject to the exponential bound (6.58).

Because all relationships, except (8.67), stated in these theorems can be satisfied by a suitable choice of control algorithm parameters, under the assumption of $\inf\limits_{t \geqslant t_0} ca_2(t) > 1$ what we have proved means that the parametrically indefinite $\mathcal{P}_x$-process is $\mathcal{A}_{\mu\nu}$ controlled for the control algorithm (8.23). Below we wish to demonstrate that in fact the $\mathcal{P}_x$-process is exponentially $\mathcal{A}_{\mu\nu}$ controlled. To achieve this goal it suffices to establish the validity of the bound (4.4a) since the fact of exponential convergence of controlled processes to O_x has been proved above. The proof of this fact makes up the subject of Problem E.

Problem E. Note that the presence of the second component $v(t)$ in the control $\boldsymbol{u}(t)$ prevents our using of the argument employed earlier in similar situations, in particular the estimate derived in Lemma 6.7-2 fails for the type of $S_{\mu\nu}$ systems examined.

We consider for $t \geqslant t_0$ a V solution $(x(t), \mu(t))$ of the $S_{\mu\nu}$ system (8.23)-(8.24) to derive a sufficient estimate from above for $\| x(t) \|$ for $t = t_1 = t_0 + 2/\alpha$. To this end, we write the integral equation for $\sigma(t) = \sigma[x(t)]$ as follows:

$$\sigma(t) = \sigma(t_0) + \int_{t_0}^{t} \left[-\frac{ca_2 - 1}{c}\sigma + c\left(a_1 - \frac{a_1}{c} + \frac{1}{c^2}\right)x + ck^0\mu|x| - c\gamma\delta \operatorname{sat}\lambda \right] d\tau \tag{8.31}$$

Under the assumption (6.67) the first term in the integrand is opposite in sign to $\sigma(t)$ and so is the last term in view of the definition of $\lambda(t)$, therefore from (8.30) we get

$$|\sigma(t) \leqslant \sigma(t_0) + \int_{t_0}^{t} c\left[\left|\left(a_1 - \frac{a_2}{c} + \frac{1}{c^2}\right)\right| + k^0\right]|x|\, d\tau \tag{8.32}$$

To estimate the right-hand side of this inequality we resort to (8.29) and observe that always $\overline{\overline{k^0}} \geqslant \widetilde{k^0}$ as can be seen from (6.96) and (6.76). As a result we get

$$|\sigma(t)| \leqslant |\sigma(t_0)| \int_{t_0}^{t} \frac{\alpha\delta}{2}|x|\, d\tau \tag{8.33}$$

This inequality can be utilized in the proof of the following auxiliary statement.

Lemma 8.2-2. Let the conditions of Theorem 8.2-2 be satisfied and for a V solution $(x(t), \mu(t))$ of the $S_{\mu\nu}$ system (8.23)-(8.24) we have $|\sigma(t)| > \delta |x(t)|$ for all $t \in [t_1, t_2]$, then

$$\|x(t)\| < c_m \frac{1 + c + \delta}{c\delta} \|x(t_1)\| \exp\left[\frac{\alpha}{2}(t - t_1)\right] \tag{8.34}$$

where $c_m = \max(c, 1)$, for the said t.

Proof. Under the conditions of the lemma for every $t \in [t_1, t_2]$ we may write

$$|\sigma(t)| \leqslant |\sigma(t_1)| + \int_{t_1}^{t} \frac{\alpha\delta}{2}|x|\, d\tau$$

which upon account of $|\sigma(t)| > \delta |x(t)|$ leads us to

$$|\sigma(t)| < |\sigma(t_1)| + \frac{\alpha}{2}\int_{t_1}^{t} |\sigma|\, d\tau$$

or with the Gronwall-Bellman lemma

$$|\sigma(t)| < |\sigma(t_1)| \exp[\alpha(t - t_1)/2], \quad t \in [t_1, t_2] \tag{8.35}$$

Now we resort to the differential constraint $c\dot{x} + x = -\sigma$ to obtain $|\dot{x}| \leqslant (|\sigma| + |x|)/c$, or, with account of $|\sigma| > \delta |x|$, $|\dot{x}| < (\delta + 1)|\sigma|/c\delta$. Because $\|x\| = |\dot{x}| + |x|$, substituting into the last inequality the bound (8.35) yields

$$\|x(t)\| < \frac{1+c+\delta}{c\delta} |\sigma(t_1)| \exp\left[\frac{\alpha}{2}(t-t_1)\right], \quad t \in [t_1, t_2]$$

Observing the ever valid inequality $|\sigma| \leqslant c_m \|x\|$ produces the desired bound (8.34). The lemma is proved.

With this lemma on hand let us continue our evaluation of a V solution on the time interval $[t_0, t_1]$, where $t_1 = t_0 + 2/\alpha$. The respective process $x(t)$ breaks this interval into two systems of alternating time intervals, namely, $\tau = \{\tau_i\}$ and $T = \{T_j\}$, where the subscripts i and j vary through some sets of integers I_τ and I_T, respectively, and where $|\sigma(t)| \leqslant \delta |x(t)|$ for $t \in \tau$, and $|\sigma(t)| > \delta |x(t)|$ for $t \in T$.

Denote by $|\tau_i|$ and $|T_j|$ the length of elementary time intervals, and $|\tau|$ and $|T|$ the respective totals, then it is clear that

$$|\tau| + |T| + \sum_{i \in I_\tau} |\tau_i| + \sum_{j \in I_T} |T_j| = 2/\alpha$$

Let also $\|x\|^{\tau_i}$ and $\|x\|_{\tau_i}$ ($\|x\|^{T_j}$ and $\|x\|_{T_j}$) stand respectively for the norms of $x(t)$ taken at the right and left ends, respectively, of these time intervals. Then for adjacent intervals $\|x\|^{\tau_i} = \|x\|_{T_i}$ and $\|x\|^{T_i} = \|x\|_{\tau_{i+1}}$. Denote also for convenience $N_1 = (1+c+\delta)/c$ and $N_2 = c_m(1+c+\delta)/c\delta$. By Theorem 6.7-1 for each $i \in I_\tau$

$$\|x\|^{\tau_i} \leqslant N_1 \|x\|_{\tau_i} \tag{8.36}$$

and by Lemma 8.2-2 for each $j \in I_T$

$$\|x\|^{T_j} < N_2 \|x\|_{T_j} \exp(\alpha T_j/2) \tag{8.37}$$

Let n_τ and n_T indicate the number of each type of time interval (obviously finite), then in view of (8.36) and (8.37) we get the following bound for $\|x(t_1)\|$:

$$\|x(t_1)\| < N_1^{n_\tau} N_2^{n_T} \|x(t_0)\| \exp(\alpha T/2)$$

which in view of $|T| \leqslant 2/\alpha$ boils down to

$$\|x(t_1)\| \leqslant N_3 \|x(t_0)\| \tag{8.38}$$

where

$$N_3 = N_1^{n_\tau} N_2^{n_T} e \tag{8.39}$$

Now we note that if for some $t_3 \in [t_1, t_2]$ $\sigma(t)$ vanishes, then subject to relationships of Theorem 8.2-2 for all $t \geqslant t_3 x(t) \in G_\delta$ and hence by Theorem 6.7-1 the bound (6.58) holds. Therefore the inequality $\| x(t) \| \leqslant N_1 \| x(t_3) \|$ holds for all $t \geqslant t_3$, but since for all $t \in [t_0, t_1]$ we have the bound (8.38) we obtain finally

$$\| x(t) \| \leqslant N_1 N_3 \| x(t_0) \|, \quad t \geqslant t_0 \tag{8.40}$$

Let now $\sigma(t)$ retain its sign in $[t_0, t_1]$. Then at $t = t_1 \mu(t_1) = -\operatorname{sgn} \sigma(t_1)$. Substituting this expression into (8.31) we convince ourselves that under the above conditions $|\sigma(t)| \leqslant |\sigma(t_1)| = c_m \| x(t_1) \|$ for all $t \geqslant t_1$ until the process appears in the domain G_0. If $x(t_1) \in G_\delta$ then for all $t \geqslant t_1$ $x(t) \in G_\delta$ and the bound (8.40) is valid again. If $x(t_1) \notin G_\delta$ then in a finite time the process achieves the boundary ∂G_δ of the domain G_δ. At this time instant t_4 $\| x(t_4) \| = |\sigma(t_4)|/\delta \leqslant \| x(t_1) \| c_m/\delta$. Since for all $t \geqslant t_4$ $x(t) \in G_\delta$, then in this case the desired bound has the form

$$\| x(t) \| \leqslant (c_m/\delta) N_1 N_3 \| x(t_0) \|, \quad t \geqslant t_0 \tag{8.41}$$

Thus we have demonstrated that in all cases there exists a constant depending on the control system characteristics only and such that the bound (4.4a) holds. Consequently, the $\mathcal{P}_x$-process concerned is steady $\mathcal{A}_{\mu\nu}$ controlled subject to the assumption (6.67). We summarize the above development in the following statement.

Theorem 8.2-3. Under the conditions of Theorems 8.2-1 and 8.2-2 the $\mathcal{P}_x$-process is exponentially $\mathcal{A}_{\mu\nu}$ controlled with the algorithm (8.23) and the bound (6.58) holds for every controlled process starting from a finite time instant.

In shorthand this result can be presented as follows: given (6.67)

$$\mathcal{P}_x \in \mathcal{S}_{\mathcal{A}_{\mu\nu}} \{\mathcal{F}_c, I_2; N, (1 - \delta)/c\} \tag{8.42}$$

where the constant N is computed as given above.

Now we turn to Problem F concerned with the evaluation of the dependence of controlled process dynamics on $\mathcal{P}_x$ parameters in the $S_{\mu\nu}$ system being examined.

Problem F. The major result we are going to establish here is as follows: whatever the $\mathcal{P}_x$ parameters, the function $\lambda[x(t)] = \sigma[x(t)]/\delta \, | x(t) |$ exponentially decays on V solutions of the $S_{\mu\nu}$ system (8.23), (8.24) after some time instant.

We have already noted that the behavior of controlled processes can be described by the equation $c\dot{x} + x = -\delta\lambda(t) \, | x(t) |$. In the Π_x parameter space this equation is described by the point $\pi^t = [1 + \delta\lambda(t) \operatorname{sgn} x(t), c]$. Hence, exponential decay of $\lambda(t)$ means that $\lim_{t \to \infty} \pi^t = \pi^0 = [1, c]$, where point π^0 corresponds to ideal motion. In terms of the Russian original of the book, this fact is interpreted as "parametric astatism". It is essential that it takes place

irrespectively of how $\mathcal{P}_x$ parameters vary. In traditional terms, this property means that with t at infinity the controlled process dynamics is independent of $\mathcal{P}_x$ parameters.

In establishing the major result of this problem we shall proceed in two stages. First we wish to obtain the largest value which $|\lambda(t)| = |\sigma(t)|/\delta |x(t)|$ can reach on V solutions of the $S_{\mu v}$ system examined. Next we establish the relation between successive maxima of the function which will yield its decay rate. To begin with, we prove the following statement.

Lemma 8.2-3. Let all conditions of Theorems 8.2-1 and 8.2-2 be met and $(\boldsymbol{x}(t), \mu(t))$ be a V solution of the $S_{\mu v}$ system (8.23)-(8.24) such that $\boldsymbol{x}(t_0) \in G_0$ and $\|\boldsymbol{x}(t_0)\| > 0$, then for all $t \geqslant t_0$ the following bound is valid for the function $\lambda(t) = \sigma(t)/\delta |\boldsymbol{x}(t)|$

$$|\lambda(t)| \leqslant \frac{\alpha}{2M_0}(1 - e^{-2M_0/\alpha}) < 1 \tag{8.43}$$

where the constant M_0 has the form

$$M_0 = \gamma c/|x(t_0)| \tag{8.44}$$

Proof. By Theorem 8.2-2 the V solution pinpointed by the lemma provides for all $t \geqslant t_0$ $|\lambda(t)| \leqslant 1$, therefore the variation of $\lambda(t)$ for $t \geqslant t_0$ can be described by an integral equation derived as equation (8.26), namely,

$$\lambda(t) = \frac{c}{\delta}\int_{t_0}^{t}[(a_1 - a_2\zeta + \zeta^2)\operatorname{sgn} x + k^0\mu - \gamma\delta\lambda/|x|]\,d\tau \tag{8.45}$$

where $\zeta = (1 + \xi)/c$, $\xi = \delta\lambda$, and it is observed that $\lambda(t_0) = 0$.

By Theorem 8.2-1 there exists a finite time $t_1 > t_0$ such that $\lambda(t_1) = 0$ and $\lambda(t) \neq 0$ for all $t \in (t_0, t_1)$. It is clear that $|\lambda(t)|$ has a maximum in $[t_0, t_1]$. We wish to estimate it from above.

First we note that $|\lambda(t)|$ can be a maximum in $[t_0, t_0 + 2/\alpha]$ as at $t = t_0 + 2/\alpha$ $\mu(t) = -\operatorname{sgn}\lambda(t)$ and the integrand in (8.45) is opposite in sign to $\lambda(t)$. Assume for definiteness $\lambda(t) > 0$ for $t \in (t_0, t_0 + 2/\alpha)$. Using (8.45) we obtain with (8.29) and $|x(t)| \leqslant |x(t_0)|$ following from (6.31) that

$$\lambda(t) \leqslant \int_{t_0}^{t}\frac{\alpha}{2}\,d\tau - \int_{t_0}^{t}\frac{c\gamma}{|x(t_0)|}\lambda\,d\tau,\quad t \in [t_0,\ t_0 + 2/\alpha] \tag{8.46}$$

Denoting $\theta = t - t_0$, $\bar{\lambda}(\theta) = \lambda(t_0 + \theta)$ carries (8.46) to

$$\bar{\lambda}(\theta) \leqslant \frac{\alpha}{2}\theta - M_0\int_0^{\theta}\bar{\lambda}\,d\theta,\quad \theta \in [0,\ 2/\alpha] \tag{8.47}$$

where M_0 is given in (8.44).

Introduce the comparison function $\lambda_c(\theta)$ as the solution of

$$\lambda_c(\theta) = \frac{\alpha}{2}\theta - M_0 \int_0^\theta \lambda_c \, d\theta, \quad \theta \in [0, 2/\alpha] \tag{8.48}$$

It is quite obvious that $\bar{\lambda}(\theta) \leqslant \lambda_c(\theta)$ for $\theta \in [0, 2/\alpha]$. The solution of this equation is $\lambda_c(\theta) = (\alpha/2M_0)[1 - \exp(-M_0\theta)]$ which is a maximum at $\theta = 2/\alpha$, therefore

$$\max_{\theta \in [0,\ 2/\alpha]} \bar{\lambda}(\theta) = \max_{t \in [t_0,\ t_0+2\alpha]} \lambda(t) \leqslant \frac{\alpha}{2M_0}(1 - e^{-2M_0/\theta})$$

It is straightforward to show that with a finite value of $|x(t_0)|$ the right-hand side of this inequality is always under unity.

To conclude the proof of the lemma we notice that for all $t \in (t_0 + 2/\alpha, t_1]$ $|\lambda(t)| < (\alpha/2M_0)(1 - e^{-2M_0/\alpha})$. For time following t_1 the argument can be repeated with account of $|x(t)| < |x(t_0)|$. The lemma is proved.

According to this lemma for every $x(t_0)$ there exists a $\lambda_0 < 1$ such that for all $t \geqslant t_0$ $x(t) \in G_{\delta\lambda_0}$ once $x(t_0) \in G_0$. Since $|x(t)|$ decreases under these conditions, then evaluating the successive instants where $\lambda(t)$ vanishes we can establish that the sequence of maxima of $|\lambda(t)|$ falls off exponentially. To realize this plan we first perform some model groundwork. Our model will be the function $\varphi(y) = (1 - e^{-y})/y$. For $y > 0$ the derivative of the function $d\varphi/dy = (1 + y - e^y)/y^2e^y$ is negative, hence $\varphi(y)$ monotonously decreases as y increases. This property carries over to the function

$$\psi_0(r) = \frac{\alpha |x(t_0)|}{2\gamma c} r \left[1 - \exp\left(-\frac{2\gamma c}{\alpha |x(t_0)|}\frac{1}{r}\right)\right] \tag{8.49}$$

which monotonously decreases as $r \to 0$ and $\psi_0(0) = 0$. Therefore for any number ρ, $0 < \rho < 1$, there is a unique solution r_ρ to the equation

$$\psi_0(r_\rho) = \rho\psi(1) \tag{8.50}$$

Let us estimate the root r_ρ of this equation from above and below. Substituting (8.49) into (8.50) yields

$$(1 - e^{-2M_0/\alpha})\rho < r_\rho < \rho \tag{8.51}$$

Denote $T = \{t_q\}_{q=0}^{\infty}$ the set of time instants where $\lambda(t)$ vanishes, and also $x_q = |x(t_q)|$, $M_q = \gamma c/x_q$, and $\psi_q(r) = (\alpha/2M_q)\, r\,(1 - e^{-2M_q/\alpha r})$ for all $q = 0, 1, \ldots, \infty$.

Finally, in similarity with (8.50) we can define the root r_ρ^q, $q = 1, 2, \ldots, \infty$ with the respective bounds as follows:

$$\psi_q(r_\rho^q) = \rho\psi_q(1) \tag{8.52}$$

$$(1 - e^{-2M_0/\alpha})\rho < r_\rho^q < \rho \tag{8.53}$$

Recall now that for $x(t) \in G_\delta = \{x: |\sigma| \leqslant \delta |x|\}$ we have the bound $|x(t)| \leqslant x_q \exp[-(1-\delta)(t-t_q)/c]$, $t \geqslant t_q$, that is, $x(t)$ decreases for $\delta \in (0, 1)$.

Let us require that

$$x_{q+1} \leqslant r_\rho^q x_q \tag{8.54}$$

For this purpose $t_{q+1} - t_q$ should be such that

$$t_{q+1} - t_q \geqslant \frac{c}{1-\delta} \ln \frac{1}{r_\rho^q} \geqslant \frac{c}{1-\delta} \ln \frac{1}{\rho[1-\exp(-2M_0/\alpha)]} \tag{8.55}$$

Here we have observed the above bounds for $|x(t)|$ and r_ρ^q. Of course, the interval $t_{q+1} - t_q$ between adjacent zeroes of $\lambda(t)$ may not satisfy the lower bound (8.55). Then from the set $T = \{t_q\}_{q=0}^\infty$ we may always take a subsequence of time instants $T' = \{t_q'\}_{q=0}^\infty$ for which (8.55) holds and apply the argument to this subsequence.

Assume for simplicity that (8.55) holds. Then it is easy to verify that for all $q \geqslant 0$ $\psi_{q+1}(1) \leqslant \psi_q(r_\rho^q) = \rho\psi_q(1)$ and, since by Lemma 8.2-3 $|\lambda(t)| \leqslant \psi_q(1)$, for $t \geqslant t_q$ we have the bound

$$|\lambda(t)| \leqslant \rho^q \psi_0(1), \quad t \geqslant t_q, \quad q \geqslant 0 \tag{8.56}$$

Letting now $\rho = \psi_0(1) < 1$ we get from (8.56)

$$|\lambda(t)| \leqslant \exp\left[\frac{\ln \psi_0(1)}{T_0}(t-t_0)\right], \quad t \geqslant t_0 \tag{8.57}$$

where $T_0 = [c/(1-\delta)] \ln [1/\psi_0(1)(1-e^{-2M_0/\alpha})]$ is determined from (8.55).

After this reasoning we are in a position to formulate the following statement.

Theorem 8.2-4. Let all conditions of Theorems 8.2-1 and 8.2-2 be satisfied and $(x(t), \mu(t))$ be a V solution of the $S_{\mu\nu}$ system (8.23)-(8.24). If $x(t_0) \in G_0$ then for all $t \geqslant t_0$ we have for the function $\lambda(t) = \sigma[x(t)]/\delta |x(t)|$ the exponential bound

$$|\lambda(t)| \leqslant \exp \frac{\ln \psi_0(1)}{T_0}(t-t_0) \tag{8.58}$$

where

$$\psi_0(1) = (\alpha |x(t_0)|/2\gamma c)[1-\exp(-2\gamma c/\alpha |x(t_0)|)] < 1 \tag{8.59}$$

$$T_0 = [c/(1-\delta)] \ln \{1/\psi_0(1)[1-\exp(-2\gamma c/\alpha |x(t_0)|)]\} \tag{8.60}$$

By way of comment on this result we note that the damping factor in the exponential decay of $|\lambda(t)|$ grows with the number $\gamma > 0$ and for a sufficiently large γ can be increased to not under a specified

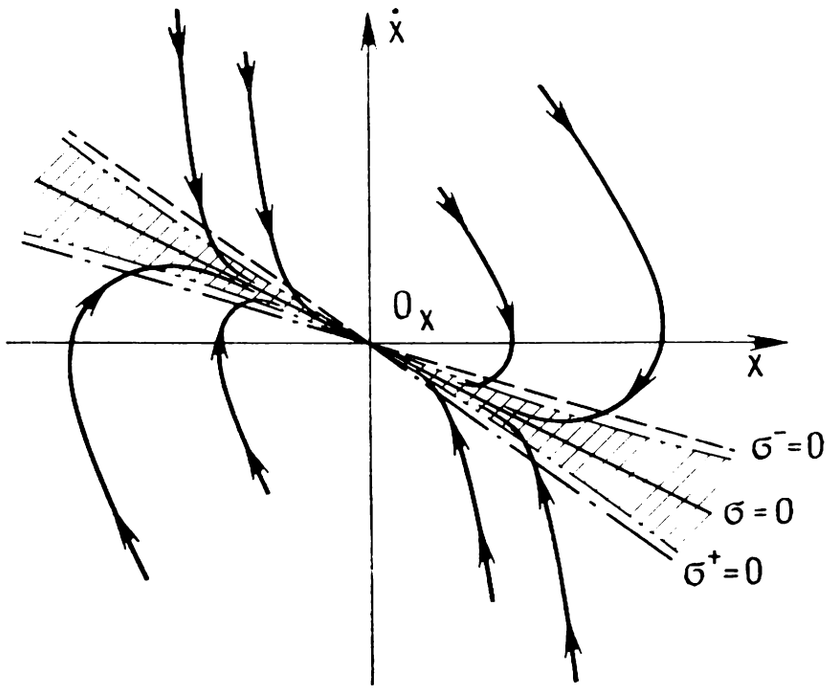

Figure 8.11

number. This can be explained as follows. For $x(t) \in G_\delta$ the second component of control (8.19) is

$$v(t) = -\gamma\delta \operatorname{sat} \lambda(t) = -\gamma\sigma(t)/|x(t)|$$
$$= (\gamma/|x(t)|)[c\dot{x}(t) + x(t)]$$

Here the function $\gamma/|x(t)|$ can be viewed as a variable gain parameter. As $t \to \infty$ $|x(t)| \to 0$, that is, this gain parameter grows without bound securing ever more accurate equality for $c\dot{x}(t) + x(t) = 0$.

So, we have demonstrated that the incorporation of a static OCFB loop into an S_μ system results in two important properties. For one thing, the $\mathcal{P}_x$-process becomes exponentially controlled, for the other, the system enjoys "parametric astatism" irrespective of how $\mathcal{P}_x$ parameters may vary, i.e. as $t \to \infty$ the dynamic properties of the controlled processes are independent of these parameters. Moreover, this asymptotic behavior can be made of any desired nature. Consequently, such $\mathcal{A}_{\mu\nu}$ algorithms enable complete solution of the control problem to be achieved.

A general insight into the nature of transients in the $S_{\mu\nu}$ system examined is provided by the projections of its V solutions onto the phase plane $(x, \dot{x})$. Figure 8.11 depicts trajectories plotted for some fixed $\mathcal{P}_x$ parameters. The hatched area represents the domain where the controlled processes remain after some time instant. The lines $\sigma^\pm = 0$ are the boundaries of the domain G_δ.

8.3 $S_{\mu\rho\nu}$ (1) Systems with Proportional Operator-Coordinate and Integral Coordinate-Operator Feedback

The block diagram of $S_{\mu\rho\nu}$ (1) systems considered in this section is given in Fig. 8.5. The equations of motion for these systems are given by the equation of the $\mathcal{P}_x$-process being controlled

$$\ddot{x}(t) + a_2(t)\,\dot{x}(t) + a_1(t)\,x(t) = u(t),\ t \geqslant t_0 \tag{8.61}$$

and the set constituting the $\mathcal{A}_{\mu\rho\nu}$ (1) algorithm

$$\begin{aligned}
\bar{u}(t) &= u(t) + v(t)\\
u(t) &= \mu(t)\,k^0\,|\,x(t)\,|\\
v(t) &= -\gamma\delta\ \mathrm{sat}\ \lambda(t)\\
\dot{\mu}(t) &= -\alpha\ \mathrm{sgn}\ \sigma'(t) \quad \text{for} \quad |\,\mu(t)\,| \leqslant 1\\
&= -\omega\mu(t) \quad \text{for} \quad |\,\mu(t)\,| > 1,\ \ |\,\mu(t_0)\,| \leqslant 1,\quad t \geqslant t_0\\
\sigma'(t) &= \sigma(t) + \delta\mu(t)\,|\,x(t)\,|\\
\sigma(t) &= -[c\dot{x}(t) + x(t)]\\
\lambda(t)\,\delta\,|\,x(t)\,| &= \sigma(t)
\end{aligned} \tag{8.62}$$

The investigation of such an $S_{\mu\rho\nu}$ system exactly parallels the development for the $S_{\mu\nu}$ system performed in the previous section, therefore here we shall content ourselves to brief comments.

First of all we note that if $x(t) \in G_\delta$ then $\mathrm{sgn}\ \sigma'(t) = \mathrm{sgn}\ \sigma(t)$ and consequently the equations of motion for the $S_{\mu\rho\nu}$ system coincide with the equations of the $S_{\mu\nu}$ system. Therefore, for the $S_{\mu\rho\nu}$ system a direct analog of Theorem 8.2-1 applies, i.e. the condition (6.67) and relationship (8.27) ensure for every controlled process in the $S_{\mu\rho\nu}$ system (8.61)-(8.62) getting into the set Γ (7.18) in finite time. The following problem presents conditions for this set to be invariant of the $S_{\mu\rho\nu}$ system.

Problem B. Consider for $t \geqslant t_0$ a V solution $(x(t),\ \mu(t))$ of the $S_{\mu\rho\nu}$ system (8.61)-(8.62). Let at $t = t_0$ $V \in \Gamma = \{(x,\ \mu)\colon \sigma'(x,\ \mu) = 0,\ |\,\mu\,| \leqslant 1\}$. Under these conditions for t posterior and close to t_0

$$\begin{aligned}
\sigma'(t) = \int_{t_0}^{t} [\dot{\sigma} + \delta\mu\,|x|^{\cdot} + \delta\,|x|\,\dot{\mu}]\,d\tau = c\int_{t_0}^{t} \Big[\Big(a_1 - a_2\frac{\xi_\lambda}{c} + \frac{\xi_\lambda\xi_\mu}{c^2}\Big)\,x\\
+ k^0\mu\,|x| - \gamma\delta\ \mathrm{sat}\ \lambda - \frac{\delta\,|x|}{c}\,\dot{\mu}\Big]\,d\tau
\end{aligned} \tag{8.63}$$

where $\xi_\lambda = 1 + \delta\lambda\ \mathrm{sgn}\ x$, and $\xi_\mu = 1 - \delta\mu\ \mathrm{sgn}\ x$.

For $|\mu(t_0)| = 1$ and $\mu(t) = -\operatorname{sgn}\sigma'(t)$ this equation becomes for $t \geqslant t_0$

$$\sigma'(t) = c\int_{t_0}^{t}\left[\left(a_1 - a_2\frac{\xi_\lambda}{c} + \frac{\xi_\lambda\xi_\mu}{c^2}\right)x - k^0|x|\operatorname{sgn}\sigma' - \gamma\delta\operatorname{sat}\lambda\right]d\tau \tag{8.64}$$

whereas for $|\mu(t_0)| < 1$ or $|\mu(t_0)| = 1$ but $|\mu(t)| < 1$ for $t > t_0$ it reads

$$\sigma'(t) = c\int_{t_0}^{t}\left[\left(a_1 - a_2\frac{\xi_\lambda}{c} + \frac{\xi_\lambda\xi_\mu}{c^2}\right)x + k^0\mu|x| - \gamma\delta\lambda - (\alpha\delta|x|/c)\operatorname{sgn}\sigma'\right]d\tau \tag{8.65}$$

The analysis of equations (8.64) and (8.65) will be carried out along the lines elaborated in Section 7.1 in the investigation of invariance for the set Γ. It can be verified that if $\|x(t_0)\| \neq 0$, then with $\alpha\delta > c(\gamma\delta + \bar{\bar{k}}^0 + k^0)$, $k^0 > \bar{k}^0$ the identity $\sigma'(t) = 0$ remains valid for t close to t_0. For this system, the fact analogous to Lemma 8.2-1 can be proved, so that the following theorem takes place.

Theorem 8.3-1. Suppose with some positive constants c and δ, $0 < \delta < 1$, the condition (6.67) is met and

$$k^0 \geqslant \max(\tilde{k}^0, \bar{k}^0) \tag{8.66}$$

$$\alpha \geqslant \frac{c}{\delta}(\gamma\delta + k^0 + \bar{\bar{k}}^0) \tag{8.67}$$

where the constants $\tilde{k}^0$, $\bar{k}^0$, and $\bar{\bar{k}}^0$ are defined in (6.94), (6.86), and (6.72). Then for any positive constant γ the set Γ (7.18) is invariant on the V solutions of the $S_{\mu\rho\nu}$ system (8.61)-(8.62).

We note again that incorporation of OFB in binary control systems enables decreasing gain values in the CFB and COFB loops. In addition, with the relations stated in Theorem 8.3-1 all the controlled processes reach the set Γ in finite time, and because in the circumstances $\sigma(t) + \delta\mu(t)|x(t)| = 0$ such controlled processes $x(t) \in G_\delta$. Therefore Theorem 6.7-1 and the bound (6.58) hold, i.e. all processes in the $S_{\mu\rho\nu}$ system converge exponentially to O_x.

The proof of the fact that controlled processes satisfy (4.4a) under the above conditions may be achieved along the lines delineated in the previous section. Therefore we go directly to formulating the associated shorthand expression

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_{\mu\rho\nu}}(1)\{\mathcal{F}_c, I_2; N, (1-\delta)/c\} \tag{8.68}$$

implying that the $\mathcal{P}_x$-process is exponentially $\mathcal{A}_{\mu\rho\nu}(1)$ controlled with the algorithm (8.62). The constant N is computed in Section 8.2.

Let us derive differential equations satisfied by controlled processes in the $S_{\mu\rho\nu}$ system when $V \in \Gamma$ for $t \geqslant t_1$. Resolving $\sigma(t) + \delta\mu(t) \mid x(t) \mid = 0$ for $\mu(t)$ to substitute it into the expression for the first component of $\underline{u}(t)$, we get $u(t) = -\sigma(t) k^0/\delta$. Observe further that in this case $\mid \mu(t) \mid \leqslant 1$ and $\mu(t) = -\lambda(t)$, therefore the second component of $\underline{u}(t)$ can be given as $v(t) = -\gamma\sigma(t)/\mid x \mid$. Substitution into the equation for $\mathcal{P}_x$-process (8.61) yields

$$\ddot{x} + \left(a_2 + \frac{ck^0}{\delta} + \frac{c\gamma}{|x|}\right)\dot{x} + \left(a_1 + \frac{k^0}{\delta} + \frac{\gamma}{|x|}\right)x = 0, \quad t \geqslant t_1 \tag{8.69}$$

This equation leads us to draw the following conclusions. First, in systems with static OCFB an operator feedback is not linearizing although produces some forcing effect. Second, the effect of large gain, whose part is not played by the sum $k^0/\delta + \gamma/\mid x(t) \mid$, also takes place in such systems. The effect is variable and increases in significance as the transient decays out. This quality ensures exponential damping of $\mu(t)$, consequently such control systems will exhibit parametrically astatic behavior. Finally, at constant $\mathcal{P}_x$ parameters operator feedback eliminates oscillations of controlled processes about the domain G_0, which have always occurred in $S_{\mu\nu}$ systems. The following problem proves the last two statements.

Problem F. We write the equation describing variation of $\lambda(t) = \sigma(t)/\delta \mid x(t) \mid$ along the V solution $(\boldsymbol{x}(t'), \mu(t))$ of the $S_{\mu\rho\nu}$ system (8.61)-(8.62) from a time t_1 when $\sigma'(t) = 0$. In this case, as noted above, the first component of the V solution satisfies equation (8.69), while the second component satisfies $\mu(t) = -\lambda(t)$, $t \geqslant t_1$. Denote $\lambda_1 = \lambda(t_1)$, then the desired equation becomes for $t \geqslant t_1$

$$\lambda(t) = \lambda_1 + \frac{c}{\delta}\int_{t_1}^{t}\left\{\left[a_1 - a_2\frac{\xi}{c} + \left(\frac{\xi}{c}\right)^2\right]\operatorname{sgn} x - \left(k^0 + \frac{\delta\gamma}{|x|}\right)\lambda\right\} d\tau \tag{8.70}$$

where $\xi(t) = 1 + \delta\lambda(t) \operatorname{sgn} x(t)$.

To determine whether controlled processes $\boldsymbol{x}(t)$ in the examined system cross the domain $G_0 = \{\boldsymbol{x}: \sigma(\boldsymbol{x}) = 0\}$ when $\lambda = 0$ it will suffice to compute the integrand in (8.70) for $\lambda = 0$. This expression is readily derived to be $(a_1 - a_2/c + 1/c^2) \operatorname{sgn} x(t)$. At fixed plant parameters the sum in parentheses has a constant sign, and the second factor is also constant for $\boldsymbol{x}(t) \in G_\delta$. Consequently, for V solutions considered the function $\lambda(t)$ changes sign at most once for $t \geqslant t_1$. This means that static OFB does eliminate oscillations of controlled processes about the domain G_0.

Now we wish to demonstrate that an exponentially decreasing majorant function can be ensured for $\lambda(t)$. Suppose for definiteness

that $\lambda_1 = \lambda(t_1) > 0$ and for $t > t_1$ $\lambda(t)$ has no sign reversals. Then using (6.72) and the bound (6.31) in (8.70) leads for $t \geqslant t_1$ to

$$\lambda(t) \leqslant \lambda_1 + \frac{c}{\delta} \int_{t_1}^{t} \left\{ \bar{\bar{k}}^0 - \left[k^0 + \frac{\gamma\delta}{|x(t_1)|} \times \exp \frac{1-\delta}{c}(\tau - t_1) \right] \lambda \right\} d\tau, \quad t \geqslant t_1 \tag{8.71}$$

Denote for convenience $\delta_c = (1 - \delta)/c$, $k^1 = \gamma\delta/|x(t_1)|$, $\theta = t - t_1$, $\lambda(\theta) = \lambda(\theta + t_1)$. Now (8.71) rewrites

$$\bar{\lambda}(\theta) \leqslant \lambda_1 + \frac{c}{\delta} \int_0^{\theta} \{\bar{\bar{k}}^0 - [k^0 + k^1 \exp(\delta_c \tau)] \bar{\lambda}\} d\tau, \quad \theta \geqslant 0 \tag{8.72}$$

Introducing the comparison function $\lambda_c(\theta)$ as

$$\lambda_c(\theta) = \lambda_1 + \frac{c}{\delta} \int_0^{\theta} \{\bar{\bar{k}}^0 - [k^0 + k^1 \exp(\delta_c \tau)] \lambda_c\} d\tau, \quad \theta \geqslant 0 \tag{8.73}$$

and subtracting it from (8.72) we get the inequality

$$\bar{\lambda}(\theta) - \lambda_c(\theta) \leqslant \frac{c}{\delta} \int_0^{\theta} [k^0 + k^1 \exp(\delta_c \tau)] (\bar{\lambda} - \lambda_c) d\tau, \quad \theta \geqslant 0$$

which yields $\lambda_c(\theta) \geqslant \bar{\lambda}(\theta)$ for all $\theta \geqslant 0$.

The comparison function $\lambda_c(\theta)$ satisfies the differential equation that follows from (8.73)

$$\dot{\lambda}_c(\theta) = \frac{c}{\delta} \bar{\bar{k}}^0 - \frac{c}{\delta} [k^0 + k^1 \exp(\delta_c \theta)] \lambda_c(\theta) \tag{8.74}$$

for all $\theta \geqslant 0$ and $\lambda_c(0) = \lambda_1$. The solution to this equation written in explicit form indicates that for $\lambda_c(\theta)$, and hence for $\lambda(t)$, there exists an exponentially decreasing majorant function. This expression, however, is extremely unwieldy and since the exact value of the majorant's exponent is not significant for our latter discussion we confine ourselves to establishing the majorant existence indirectly. To this end, we change the variable in (8.74) as follows:

$$\lambda_c(\theta) \exp(\delta_c \theta) = z(\theta) + \bar{\bar{k}}^0 / k^1 \tag{8.75}$$

Straightforward calculations (details suppressed) show that for $\theta \geqslant 0$ the function $z(\theta)$ satisfies the differential equation

$$\dot{z} = -\frac{ck^1}{\delta} e^{\delta_c \theta} z - \left(\frac{ck^0}{\delta} - \delta_c \right) \left(z + \frac{\bar{\bar{k}}^0}{k^1} \right) \tag{8.76}$$

subject to the initial condition $z(0) = \lambda_1 - k^0/k^1$.

Assume now that

$$(c/\delta)\, k^0 \geqslant \delta_c = (1 - \delta)/c \tag{8.77}$$

Then the addend on the right-hand side of equation (8.76) is always nonpositive. Therefore, if $x(0) \geqslant 0$ then $z(\theta) \leqslant z(0)$ for all $\theta \geqslant 0$; if on the other hand $z(0) < 0$ then $z(\theta) < 0$ for all $\theta > 0$. Recalling that $z(0) = \lambda_1 - \bar{\bar{k}}^0/k^1$ and $\lambda_1 \geqslant 1$ the inequality in $z(\theta)$ can be rewritten as

$$z(\theta) \leqslant \max[0,\ 1 - \bar{\bar{k}}^0/k^1],\quad \theta \geqslant 0 \tag{8.78}$$

Substituting this inequality into (8.76) with account of $\theta = t - t_1$ produces the desired bound

$$|\lambda(t)| \leqslant \max(\bar{\bar{k}}^0/k^1,\ 1) \exp[-\delta_c(t - t_1)],\quad t \geqslant t_1 \tag{8.79}$$

Theorem 8.3-2. Let all the conditions of Theorem 8.3-1 and (8.77) be satisfied, then for every V solution $(x(t),\ \mu(t))$ of the $S_{\mu\rho\nu}$ system (8.61)-(8.62) there exists a finite time $t_1 \geqslant t_0$ such that for all $t \geqslant t_1$

$$\lambda(t) = \frac{\sigma(t)}{\delta |x(t)|} \leqslant \max(1,\ \bar{\bar{k}}^0/k^1) \exp\left[-\frac{1-\delta}{c}(t - t_1)\right] \tag{8.80}$$

where $\bar{\bar{k}}^0$ is defined in (6.72), and

$$k^1 = \gamma\delta/|x(t_1)| \tag{8.81}$$

Thus we have demonstrated that in $S_{\mu\rho\nu}$ systems (8.61)-(8.62), when the conditions (8.67) are satisfied all controlled processes $x(t)$ converge to the origin exponentially, and their dynamic behavior becomes ever less dependent on $\mathcal{P}_x$ parameters with time. In the limit as t is at infinity such control systems exhibit parametrically astatic behavior. It is essential that this fact takes place for any variation of $\mathcal{P}_x$ parameters (admissible by problem statement) over a wide range. Also the control action is continuous and all controller gains are limited.

A qualitative insight into the character of transients in such $S_{\mu\rho\nu}$ systems can be provided by Fig. 8.12. It shows the projections of V solutions $(x(t),\ \mu(t))$ on the phase plane obtained with some fixed values of $\mathcal{P}_x$ parameters. The hatched area represents the domain with the boundary $|\sigma(x)| = kx^2$ for some $k > 0$. The lines $\sigma^- = 0$ and $\sigma^+ = 0$ define the boundaries of G_δ.

It should be noted that feedback in the considered type of control systems never goes beyond the class of admissible feedback and all requirements contained in the definition of complete solution of the control problem can be satisfied by a suitable choice of circuit parameters.

To conclude our evaluation of various binary control systems with proportional OCFB a few remarks are in order on the applicability of other COFB laws in controlling $\mathcal{P}_x$ processes or S systems. Recall that S_μ systems considered in Chapter 6 have been augmented by inertial COFB, integral COFB with variable integration rate, and

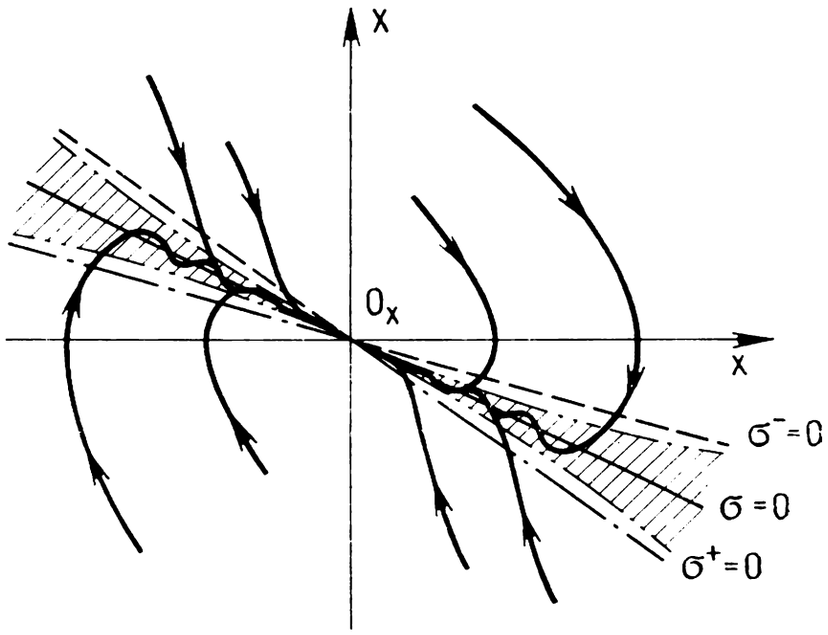

Figure 8.12

proportional OCFB. In all resultant systems, the properties of controlled processes alter as has been demonstrated in Section 8.2 for $S_{\mu\nu}$ systems with integral COFB. More specifically, from a certain time instant (individual for each process) all controlled processes exponentially converge to zero subject to the estimate (6.58); their dynamic behavior at infinite t is independent of $\mathcal{P}_x$ parameters, i.e. all such binary control systems exhibit parametrically astatic behavior when the conditions (6.67) are met. Changing from $\mathcal{A}_{\mu\nu}$ (1) to $\mathcal{A}_{\mu\nu}$ (2) control algorithms retains the key properties of $S_{\mu\nu}$ (1) system for resultant systems, the difference being that the $S_{\mu\nu}$ (2) systems no longer need satisfying the condition (6.67) thus expanding the class of $\mathcal{P}_x$-processes controlled in this manner.

An interesting effect can be observed in $S_{\mu\rho\nu}$ systems with quasicontinuous generation of control function. Whereas in $S_{\mu\rho}$ systems with quasicontinuous control algorithms the control problem is solved only Δ-accurately, the incorporation of a proportional OCFB loop enables this problem to be solved asymptotically and ensure asymptotic control with respect to parametric disturbances. All these properties can be verified with the analytical methodology developed above for binary control systems.

The findings of this section in full measure apply to the case of S system control with the difference that here parametrically astatic behavior is inachievable unless the conditions (6.182)-(6.183) are satisfied. At infinite time, in such binary control systems, the dependence of process dynamics upon S system parameters asymptotically approaches the specified function defined by equation (6.180).

To summarize, in all cases proportional OCFB proved helpful in achieving the required control performance.

8.4 $S_{\mu\nu}$ (1) Systems with Integral Laws of Coordinate-Operator and Operator-Coordinate Feedback

This section presents a qualitative investigation of the $S_{\mu\nu}$ (1) system schematized by the block diagram in Fig. 8.8. This system employs integral control laws in COFB and OCFB loops. The $\mathcal{A}_{\mu\nu}$ (1) control algorithm used is described by

$$\begin{aligned}
&\bar{u}(x,\ t) = u(x,\ t) + v(t)\\
&u(x,\ t) = \mu(t)\, k^0\, |x|\\
&v(t) \quad = \gamma\, |x(t)|\, l(t)\\
&\dot{\mu}(t) \quad = -\alpha \operatorname{sgn} \sigma(t) \quad \text{for } |\mu(t)| \leqslant 1\\
&\qquad\quad = -\omega\mu(t) \quad \text{for } |\mu(t)| > 1,\ \ |\mu(t_0)| \leqslant 1,\ \ t \geqslant t_0\\
&\dot{l}(t) = -\gamma_1 \operatorname{sat} \lambda(t) \text{ for } |l(t)| \leqslant 1\\
&\qquad = -\omega_1 l(t) \text{ for } |l(t)| > 1,\ \ |l(t_0)| \leqslant 1,\ \ t \geqslant t_0\\
&\sigma(t) = -c\dot{x}(t) - x(t)\\
&\lambda(t) = \sigma(t)/\delta\, |x(t)|
\end{aligned} \tag{8.82}$$

This set along with the $\mathcal{P}^0_x$-process equation

$$\ddot{x}(t) + a_2(t)\, \dot{x}(t) + a_1(t)\, x(t) = -\bar{u}[x(t),\ t], \quad t \geqslant t_0 \tag{8.83}$$

where $a_1(t)$ and $a_2(t)$ satisfy (4.2), represent the equations of motion for the $S_{\mu\nu}$ system under investigation.

The major goal of this study is to test $\mathcal{A}_{\mu\nu}$ algorithms (8.82) for attainability of the complete solution to the control problem for ill-defined $\mathcal{P}^0_x$-processes. Note that in such control systems the feedback $\bar{u}(x,\ t)$ relates to the class of admissible feedback at finite values of coefficients α and γ_1.

The dynamic $S_{\mu\nu}$ system under consideration is a fourth order system. For $t \geqslant t_0$ its solutions are functional sets $(x(t),\ \mu(t),\ l(t))$. The initial conditions for $\mu(t)$ and $l(t)$ select among all possible $S_{\mu\nu}$ system solutions those belonging to the set

$$V_\gamma = \{(x,\ \mu,\ l):\ x \in \mathbb{R}^2,\ \ |\mu| \leqslant 1,\ \ |l| \leqslant 1\} \tag{8.84}$$

Henceforth we shall refer to such solutions as V_γ solutions. At $\gamma = 0$ the operator-coordinate feedback circuit is open and V_γ solutions become ordinary V solutions of S_μ system.

The principal properties of V_γ solutions, including the definition of solution, its existence and continuity onto the entire semiaxis $t \geqslant t_0$, have been treated in Ref. 93. Some comments will be in order for those V_γ solutions where $x(t_0) = 0$. With this initial con-

dition imposed on $x(t)$ the value $\lambda(t_0)$ is either indefinite or infinite. In the first case $\dot{x}(t_0) = 0$ and then, as can be readily seen from (8.82) and (8.83), the respective V_γ solution has the form $(O_x, \mu(t), l(t))$ for all $t \geqslant t_0$, where $\mu(t)$ and $l(t)$ are some absolutely continuous functions such that $|\dot{\mu}(t)| \leqslant \alpha$, $|\dot{l}(t)| \leqslant \gamma_1$; this type of solutions corresponds to the definition of Ref. 93. In the second case $|\dot{x}(t_0)| > 0$ and by virtue of (8.82) and (8.83) for t posterior and close to t_0 such solution is an ordinary V_γ solution at a finite value of $\lambda(t)$. After these remarks in our further elaboration we confine ourselves to analysis of those V_γ solutions for which $x(t_0) \neq O_x$. As before, the qualitative investigation will proceed along the lines of solving Problems A to F.

Consider first conditions ensuring that the domain $G_\delta = \{x: |\sigma(x)| \leqslant \delta |x|\}$ is a domain of attraction.

Problem C. Let $(x(t), \mu(t), l(t))$ be some V_γ solution of the system under investigation. Assume $x(t) \notin G_\delta$ on this solution for all $t \geqslant t_0$. Then, as can be seen from the equation of variation of $\mu(t)$ and $l(t)$, for $\gamma_2 > 0$ and $t \geqslant t_0 + \Delta t$ with $\Delta t \leqslant \max(2/\alpha, 2/\gamma_1)$,

$$\mu(t) = -\operatorname{sgn}\sigma(t) \quad \text{and} \quad l(t) = -\operatorname{sgn}\sigma(t)$$

Substituting these expressions into the $S_{\mu\nu}$ system's dynamic equation we find that for $t \geqslant t_0 + \Delta t$ the respective process satisfies the differential equation

$$\ddot{x}(t) + a_2(t)\dot{x}(t) + a_1(t)x(t) = (k^0 + \gamma)|x(t)| \operatorname{sgn}\sigma[x(t)] \tag{8.85}$$

This equation describes a VS system similar to that of (6.65) therefore in this case we may establish direct analogs of Theorems 6.7-3 and 6.7-4. The respective result is formulated below as a corollary.

Corollary 8.4-1. Suppose with some positive constants c and δ the condition (6.67) is satisfied, γ_1 is a positive constant, and

$$k^0 + \gamma > \overline{\overline{k}}^0 \tag{8.86}$$

where $\overline{\overline{k}}^0$ is defined in (6.72). Then all controlled processes in the $S_{\mu\nu}$ system (8.82)-(8.83) divide themselves into two families of $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. For x^{I} processes there exists a finite time instant t_1 when $x(t_1) \in G_0$, whereas for x^{II} processes $\lim_{t\to\infty} \| x^{\mathrm{II}}(t) \| = 0$ and from some time instant $x^{\mathrm{II}}(t) \notin G_\delta$.

Thus the incorporation of integral operator-coordinate feedback (8.21) instead of proportional one again has brought about controlled processes asymptotically tending to the origin O_x as t goes to infinity. The properties of such controlled processes have been ela-

borated earlier. The distinctive features of this type of OCFB may manifest themselves in the properties of x^1 processes. Below we are going to reveal conditions ensuring $x(t) \in G_\delta$ for such controlled processes (henceforth we omit the superscript).

Problem B. Consider a V_1 solution $(x(t), \mu(t), l(t))$ of the $S_{\mu\nu}$ system (8.82)-(8.83) for $t \geqslant t_0$ in the assumption of $x(t_0) \in G_0$ and $\|x(t_0)\| > 0$. Under these conditions the variation of $\lambda(t)$ for $t \geqslant t_0$ can be described as

$$\lambda(t) = \frac{c}{\delta} \int_{t_0}^{t} [(a_1 - a_2\zeta + \zeta^2) \operatorname{sgn} x + \mu k^0 + l\gamma] \, d\tau \tag{8.87}$$

where $\zeta = (1 + \xi)/c$ and $\xi(t) = \delta\lambda(t) \operatorname{sgn} x(t)$.

This equation can be analyzed along the lines of the argument elaborated in the proofs of Theorems 6.7-2 and 6.9-1 to establish the following statement for solving Problem B.

Theorem 8.4-1. Let at some positive constants c and δ one of the following sets be satisfied

for any γ

$$k^0 > |\gamma| + \bar{k}^0 \tag{8.88}$$

$$\alpha \geqslant \frac{2c}{\delta}(|\gamma| + k^0 + \bar{\bar{k}}^0) \tag{8.89}$$

for any k^0 and $\alpha > 0$

$$\gamma > |k^0| + \bar{k}^0 \tag{8.90}$$

$$\gamma_1 \geqslant \frac{8c}{\delta}(\gamma + |k^0| + \bar{\bar{k}}^0) \tag{8.91}$$

or any positive γ

$$k^0 + \gamma > \bar{k}^0 \tag{8.92}$$

$$\alpha \geqslant \frac{2c}{\delta}(\gamma + k^0 + \bar{\bar{k}}^0) \tag{8.93}$$

$$\gamma_1 = 4\alpha \tag{8.94}$$

where the constants $\bar{k}^0$ and $\bar{\bar{k}}^0$ are defined in (6.86) and (6.72). Then the domain G_δ is G_0 invariant for the $S_{\mu\nu}$ system (8.82)-(8.83).

It will be noticed that the last set in the theorem conditions allows lower controller gain parameters than the other sets do.

Under the conditions of Corollary 8.4-1 and Theorem 8.4-1 two types of behavior are feasible for the controlled process, namely, it either converges to O_x or belongs to G_δ from some time instant. If in addition the conditions of Theorem 6.7-1 and Corollary 6.7-1 are met, then all controlled processes in the $S_{\mu\nu}$ system (8.82)-(8.83) converge to O_x, i.e., the $\mathcal{P}_x$-process is $\mathcal{A}_{\mu\nu}$ controlled subject to (6.67).

For this control system we can readily obtain an analog of Lemma 6.7-2 with the bound (6.74) incorporating, however, other

constants P and α. Specifically, α must satisfy one of the conditions in Theorem 8.4-1, while P is now defined as

$$P = \max_{|\xi| \leqslant 1} \{\sup_{t \geqslant t_0} |a_1(t) + \xi k^0 + \gamma|, \sup_{t \geqslant t_0} [1 + |a_2(t)|]\}$$

where k^0 must also be chosen in accord with Theorem 8.4-1. Therefore we are in a position to declare that the $\mathcal{P}_x$-process is steady $\mathcal{A}_{\mu\rho}$ controlled with the algorithm (8.82) subject to the condition (6.67). In shorthand this statement reads

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_{\mu\nu}} \{\mathcal{F}_c, I_2; N\} \tag{8.95}$$

where $N = (c_m/\delta)[(1 + \delta + c)/c]^2 \exp(2P/\alpha)$.

Now we turn to the investigation of the dependence of process dynamics in such a system on $\mathcal{P}_x$-parameters.

Problem F. We wish to determine conditions ensuring that $|\lambda(t)|$ asymptotically goes to the origin. In terms of this book, the respective control system exhibits parametrically astatic behavior.

Consider a V_γ solution $(x(t), \mu(t), l(t))$ of the $S_{\mu\nu}$ system (8.82)-(8.83) for $t \geqslant t_0$ and assume that $x(t_0) \in G_0$ and $\|x(t)\| > 0$. Then under the conditions of Theorem 8.4-1 the function $\lambda(t)$ is defined for all $t \geqslant t_0$, satisfies $|\lambda(t)| \leqslant 1$ there, and its variation with time is described by the integral equation (8.87) with $\mu(t)$ and $l(t)$ satisfying (8.82). From (8.87) we go over to the differential equation. Observing that $x(t)$ has no sign reversal for $x(t) \in G_\delta$ we get for $t \geqslant t_0$

$$\dot{\lambda} = \frac{c}{\delta}(a_1 - a_2\varkappa + \varkappa^2) \operatorname{sgn} x + \mu k^0 + l\gamma \tag{8.96}$$

where $\varkappa = (1 + \delta\lambda \operatorname{sgn} x)/c$.

In addition to this equation we write with account of $\operatorname{sgn} \sigma(t) = \operatorname{sgn} \lambda(t)$ the equations for variations of $\mu(t)$ and $l(t)$, viz.,

$$\begin{aligned} \dot{\mu}(t) &= -\alpha \operatorname{sgn} \lambda \quad \text{for } |\mu| \leqslant 1 \\ &= -\omega\mu \qquad \text{for } |\mu| < 1, \quad |\mu(t_0)| \leqslant 1, \quad t \geqslant t_0 \end{aligned} \tag{8.97}$$

$$\begin{aligned} \dot{l}(t) &= -\gamma_1\lambda \quad \text{for } |l| \leqslant 1 \\ &= -\omega_1 l \quad \text{for } |l| > 1, \quad |l(t_0)| \leqslant 1, \quad t \geqslant t_0 \end{aligned} \tag{8.98}$$

Equations (8.96)-(8.98) define a closed-loop dynamic system which will be examined to establish that $\lambda(t)$ converges to the origin. In what follows we assume γ and γ_1 are positive.

Three remarks are worth making about equation (8.96). Firstly, under some conditions $\lambda(t)$ is a sign variable function. To illustrate, assume it does not change sign, then in a time $2/\alpha$ there should be $\mu(t) = -\operatorname{sgn} \lambda(t)$. If $k_0 > \bar{\bar{k}}^0 + \gamma$, where $\bar{\bar{k}}^0$ is given in (6.72), then for all stated t $\dot{\lambda}(t) \operatorname{sgn} \lambda(t) < 0$ and hence $|\lambda(t)|$ decrease

at a finite rate. Consequently, $\lambda(t)$ is sign variable.

Secondly, if $|\mu(t)| = 1$ on a time interval $[t_1, t_2]$, $t_2 \geqslant t_1$, and $|\mu(t)| < 1$ for $t < t_1$, then with $k^0 > \bar{\bar{k}}^0 + \gamma$ for all $t \in [t_1, t_2]$ we have $\dot{\lambda}(t) \operatorname{sgn} \lambda(t) < 0$. This fact immediately follows from (8.96) and (8.97).

Finally, under the above conditions and with $2/\alpha + 1/(k^0 - \bar{\bar{k}}^0 - \gamma) < 2/\gamma_1$ we have $|l(t)| < 1$ at least for all t following a second zero of $\lambda(t)$. Let t_1 denote the closest to t_0 time instant when $\lambda = 0$. Assuming that all the aforestated relationships hold true and $\mathcal{P}_x$ parameters a_1 and a_2 are fixed within (4.2) we differentiate equation (8.96) with respect to time. For $t \geqslant t_1$ we then obtain (recall that $\operatorname{sgn} x(t) = \text{constant}$)

$$\ddot{\lambda} + \left(a_2 - \frac{2}{c} - \frac{2\delta\lambda}{c} \operatorname{sgn} x\right)\dot{\lambda} + \frac{c}{\delta}\gamma\gamma_1\lambda - \frac{c}{\delta}k^0\dot{\mu} = 0 \tag{8.99}$$

To analyze this equation we invoke the Lyapunov function

$$v(\lambda, \dot{\lambda}) = \frac{1}{2}\left(\dot{\lambda}^2 + \frac{c}{\delta}\gamma\gamma_1\lambda^2\right) + \frac{c}{\delta}\alpha k^0 \int_0^{\lambda} \operatorname{sgn} \eta \, d\eta \tag{8.100}$$

The derivative of this function with account of (8.99) is as follows:

$$\dot{v} = -\left(a_2 - \frac{2}{c} - \frac{2\delta\lambda}{c} \operatorname{sgn} x\right)\dot{\lambda}^2 + \frac{c}{\delta}\alpha k^0 \dot{\lambda} \operatorname{sgn} \lambda + \frac{c}{\delta}k^0\dot{\lambda}\dot{\mu} \tag{8.101}$$

If within some time interval $|\mu(t)| = 1$, then $\dot{\mu} = 0$ and

$$\dot{v} = -\left(a_2 - \frac{2}{c} - \frac{2\delta\lambda}{c} \operatorname{sgn} x\right)\dot{\lambda}^2 + \frac{c}{\delta}\alpha k^0 \dot{\lambda} \operatorname{sgn} \lambda \tag{8.102}$$

where $\dot{\lambda} \operatorname{sgn} \lambda < 0$ as has been proved above.

If $|\mu(t)| < 1$ on some time interval, then

$$\dot{v} = -\left(a_2 - \frac{2}{c} - \frac{2\delta\lambda}{c} \operatorname{sgn} x\right)\dot{\lambda}^2 \tag{8.103}$$

Therefore, given

$$ca_2 > 2(1 + \delta) \tag{8.104}$$

the derivative $\dot{v}(\lambda, \dot{\lambda})$ is nonpositive for all $t \geqslant t_1$. Since the set $\dot{\lambda} = 0$ contains no bounded trajectories except the point $(\lambda = 0, \dot{\lambda} = 0)$, then by Theorem 5.2 of Ref. 96 the origin of the $(\lambda, \dot{\lambda})$ space is asymptotically stable in the large. It is worth noting that equilibrium values μ^* and l^* of the variables $\mu(t)$ and $l(t)$ are related by the equation $(a_1 - a_2/c + 1/c^2) \operatorname{sgn} x(t_1) + \mu^* k^0 + l^*\gamma = 0$, suggesting that these values are independent of $\mathcal{P}_x$ parameters and

cannot be explicitly expressed through system parameters. This finding leads us to the following lemma.

Lemma 8.4-1. Let at some positive constants c, δ, γ, and γ_1 there hold the following relations $0 < \delta < 1$,

$$k^0 > \gamma + \overline{\overline{k}}^0 \tag{8.105}$$

with $\overline{\overline{k}}^0$ from (6.72),

$$\frac{2}{\gamma_1} > \frac{2}{\gamma} + \frac{1}{k^0 - k^0 - \gamma} \tag{8.106}$$

$$\min_{a_2} ca_2 > 2(1+\delta) \tag{8.107}$$

Then for every V_γ solution $(x(t), \mu(t), l(t))$ of the $S_{\mu\nu}$ system (8.82)-(8.83) such that $x(t_0) \in G_0$ at $\mathcal{P}_x$ parameters a_1 and a_2 fixed within (4.2)

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} \sigma[x(t)]/\delta\,|x(t)| = 0$$

and always $\lim_{t\to\infty} \|x(t)\| = 0$.

Thus we have demonstrated that under some additional conditions a portion of controlled processes (x^1 processes) in the $S_{\mu\nu}$ system (8.82)-(8.83) can be made asymptotically independent of $\mathcal{P}_x$ parameters when these take up fixed values. The principal constraint in this case is inequality (8.107) which can be treated as a stronger modification of (6.67). This inequality defines the rate of decay for $\lambda(t)$ therefore this index of the control system cannot be chosen at will. From this aspect, proportional OCFB seems a preferable proposition.

A qualitative insight into the behavior of controlled processes in this $S_{\mu\nu}$ system can be obtained from Fig. 8.13.

8.5 $S_{\mu\rho\nu}(1)$ Systems with Integral Laws of Coordinate-Operator and Operator-Coordinate Feedback

The block diagram of the control system to be evaluated in this section is presented in Fig. 8.9. The respective $\mathcal{A}_{\mu\rho\nu}(1)$ control algorithm is as follows:

$$\begin{aligned} \bar{u}(x, t) &= u(x, t) + v(t) \\ u(x, t) &= \mu(t)\, k^0\, |x| \\ v(t) &= \gamma\, |x(t)|\, l(t) \end{aligned} \tag{8.108}$$

$$\mu(t) = -\alpha \operatorname{sgn} \sigma'(t) \quad \text{for } |\mu(t)| \leqslant 1$$
$$= -\omega\mu(t) \quad \text{for } |\mu(t)| > 1, \quad |\mu(t_0)| \leqslant 1, \quad t \geqslant t_0$$

$$\dot{l}(t) = -\gamma_1 \operatorname{sat} \lambda(t) \quad \text{for } |l(t)| \leqslant 1$$
$$= -\omega_1 l(t) \quad \text{for } |l(t)| > 1, \quad |l(t_0)| \leqslant 1, \quad t \leqslant t_0$$

$$\sigma' = \sigma + \delta\mu |x|, \quad \sigma = -(c\dot{x} + x), \quad \lambda = \sigma/\delta |x|$$

This set along with the $\mathcal{P}_x$-process equation (8.83) constitutes the equation of motion for the $S_{\mu\rho\nu}$ (1) system under examination.

The differential equations in (8.108) indicate that here we content ourselves with evaluating $S_{\mu\rho\nu}$ (1) system's solutions ($x(t)$, $\mu(t)$, $l(t)$) subject to initial conditions from the set (8.84), i.e. V_γ solutions. An elaborate investigation into V_γ solution properties in the case of the $S_{\mu\rho\nu}$ system (8.83), (8.109) could be carried out in much the same way as we did for other types of control system. To save space we limit ourselves here to elucidating some key statements and giving the associated brief comments.

Consider the set $\Gamma = \{(x, \mu): \sigma'(x, \mu) = 0, |\mu| \leqslant 1\}$. The conditions for this set to be a domain of attraction for the V solutions are formulated below.

Corollary 8.5-1. Let with some positive constants c, δ, and γ_1 the condition (6.67) be met and

$$k^0 + \gamma > \bar{k}^0 \tag{8.109}$$

where $\bar{k}^0$ is given in (6.84). Then the set Γ of (7.18) is attracting for V_γ solutions ($x(t)$, $\mu(t)$, $l(t)$) of the $S_{\mu\rho\nu}$ system (8.83), (8.108); all controlled processes divide themselves into two families: $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. For x^{I} processes there exists a finite time t_1 such that $(x^{\mathrm{I}}(t_1), \mu(t_1)) \in \Gamma$, whereas for x^{I} processes $\lim\limits_{t\to\infty} \| x^{\mathrm{II}}(t) \| = 0$ and from some finite t_2 $x^{\mathrm{II}}(t) \notin G_\delta$.

Conditions for the set Γ of (7.18) to be invariant for V_γ solutions of the $S_{\mu\rho\nu}$ system are given in the following theorem.

Theorem 8.5-1. Let the conditions of Corollary 8.5-1 be satisfied and

$$\alpha \geqslant (c/\delta)(\gamma + k^0 + \bar{\bar{k}}^0) \tag{8.110}$$

where $\bar{\bar{k}}^0$ is given in (6.72). Then for any positive constant γ the set Γ of (7.18) is invariant for V_γ solutions of the $S_{\mu\rho\nu}$ system (8.83), (8.109).

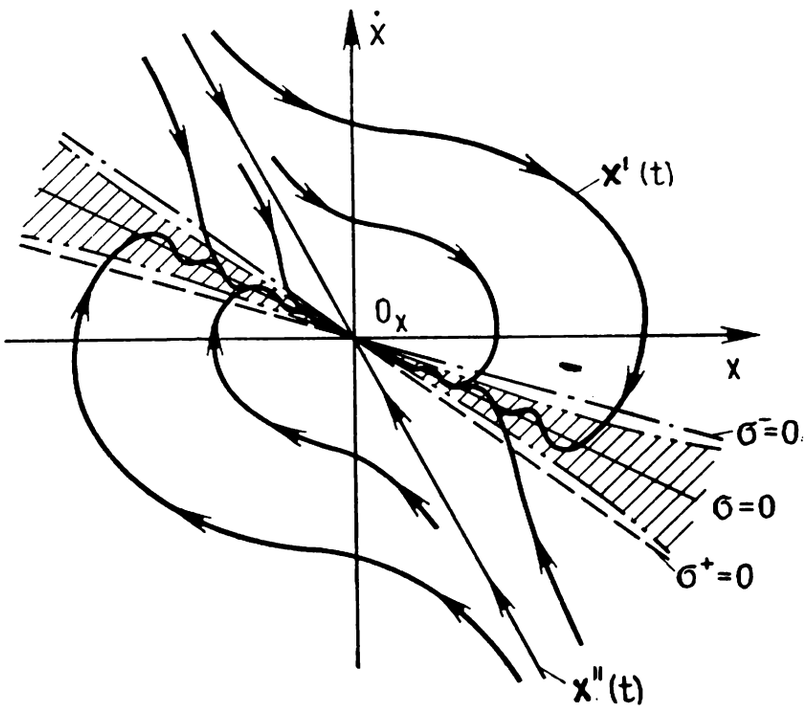

Figure 8.13

To prove this theorem it is sufficient to analyze the expression for $\sigma'(t)$ which for $t \geqslant t_0$ has the form

$$\sigma'(t) = c \int_{t_0}^{t} \left[\left(a_1 - a_2 \frac{\xi_\lambda}{c} + \frac{\xi_\lambda \xi_\mu}{c^2} \right) x + k^0 \mu |x| + \gamma |x| - \frac{\alpha\delta |x|}{c} \operatorname{sgn} \sigma' \right] d\tau \qquad (8.111)$$

where $\xi_\lambda = 1 + \delta\lambda \operatorname{sgn} x$, $\xi_\mu = 1 + \delta\mu \operatorname{sgn} x$.

Under the conditions of Theorem 8.5-1 the set Γ is attracting and invariant, therefore for x^{I} processes from some time instant $\sigma'[x(t), \mu(t)] = \sigma[x(t)] + \delta\mu(t) |x(t)| = 0$. Since for V_γ solutions there is always $|\mu(t)| \leqslant 1$, this equation is equivalent to $x(t) \in G_\delta$, hence the conditions of convergence for such processes to the origin subject to the bound (6.58) are given by Theorem 6.7-1 and Corollary 6.7-1.

The fact that the $S_{\mu\rho\nu}$ system (8.83), (8.109) is stable in the phase variables $(x, \dot{x})$ can be established in the standard manner. The resultant shorthand expression summarizing the findings of this development is as follows:

$$\mathscr{P}_x \in \mathscr{C}_{\mathscr{A}_{\mu\rho\nu}(1)} \{\mathscr{F}_c, I_2; N\} \qquad (8.112)$$

subject to the constraint (6.67), with the constant $N = (c_m/\delta) \times [(1 + \delta + c)/c]^2 \exp(2P/\alpha)$, satisfying (8.110), and P the same as in (8.95).

To complete our qualitative investigation of the $S_{\mu\rho\nu}$ system (8.83), (8.109) we are to evaluate those features in its behavior that appear after the incorporation of operator feedback. It is quite obvious that in this case only those V_γ solutions should be examined for which $\sigma'(t) = 0$. Assume for convenience that this equality holds beginning from t_0. Then resolving this equation for $\mu(t)$ and substituting the result in the equation for $u(x, t)$ in (8.109) we obtain for $t \geqslant t_0$

$$u[x(t), t] = -(k^0/\delta)\,\sigma[x(t)] = (k^0/\delta)\,[c\dot{x}(t) + x(t)] \tag{8.113}$$

Eliminating $|x(t)|$ between the equation $\sigma[x(t)] + \delta\mu(t) \times |x(t)| = 0$ and the equation for $v(t)$ in (8.109) we obtain for OCFB loop control

$$v(t) = -\frac{\gamma}{\delta}\frac{l(t)}{\mu(t)}\sigma[x(t)] = \frac{\gamma}{\delta}\frac{l(t)}{\mu(t)}[c\dot{x}(t) + x(t)] \tag{8.114}$$

having sense for $\mu(t) \neq 0$, which here was tacitly assumed. Adding up (8.113) and (8.114) we get

$$\bar{u}(t) = \left[\frac{k^0}{\delta} + \frac{\gamma}{\delta}\frac{l(t)}{\mu(t)}\right][cx(t) + x(t)], \quad t \geqslant t_0 \tag{8.115}$$

Approaching the expression in the first brackets as a sort of gain for the linear combination of the phase variables we see that in this case OFB produces a forcing effect, but, in general, is no longer a linearizing feedback as the equality $l(t) = \mu(t)$ may not hold. In addition, δ can be taken far less than unity, therefore OFB may be said to produce a large gain effect.

It should be noted that given $l(t) = \mu(t)$, $\bar{u} = [(k^0 + \gamma)/\delta] \times (c\dot{x} + x)$ and the processes considered are described by a linear differential equation. As has been already mentioned, in this case there is no asymptotic independence of controlled processes of $\mathcal{P}_x$ parameters, but with constant plant parameters (see Section 7.1) parametrically independent convergence to the origin takes place. Consequently, asymptotic independence of parametric disturbances will take place for the considered $x(t)$ when the ratio $l(t)/\mu(t)$ grows to infinity with t, and this is the case only if $\lim_{t\to\infty} \mu(t) = -\lim_{t\to\infty} \lambda(t) = 0$. This statement will be proved below.

Let us write a differential equation describing the variation of $\lambda(t) = \sigma(t)/\delta\,|x(t)|$ on the V_γ solution being considered ($x(t)$,

$\mu(t)$, $l(t)$). Since in this case for $t \geqslant t_0$ $\sigma'(t) = 0$, then $\mu(t) = -\lambda(t)$ so that

$$\dot{\lambda} = \frac{c}{\delta}\left\{\left[a_1 - a_2\frac{1+\delta\lambda\operatorname{sgn}x}{c} + \left(\frac{1+\delta\lambda\operatorname{sgn}x}{c}\right)^2\right]\operatorname{sgn}x - \lambda k^0 + l\gamma \right. \tag{8.116}$$

Similar to the development of the previous section assume that $\mathcal{P}_x$ parameters a_1 and a_2 are arbitrary but fixed. Then time differentiation of (8.116) yields the differential equation of second order valid almost for all $t \geqslant t_0$, viz.,

$$\ddot{\lambda} + \left[a_2 - \frac{2}{c}(1+\delta\lambda\operatorname{sgn}x) + \frac{c}{\delta}k^0\right]\dot{\lambda} - \frac{c}{\delta}\gamma\dot{l} = 0 \tag{8.117}$$

where $\dot{l}$ is defined in (8.109).

The fact that equation (8.117) is valid almost everywhere on $[t_0, \infty)$ follows from the function $\mu(t) = -\lambda(t)$ being absolutely continuous and its derivative coinciding with the continuous derivative $\dot{\lambda}$ only almost everywhere on $[t_0, \infty)$.

Analysis of equation (8.117) will be carried out with the Lyapunov function

$$v(\lambda, \dot{\lambda}) = \left(\dot{\lambda}^2 + \gamma\gamma_1\frac{c}{\delta}\lambda^2\right)/2 \tag{8.118}$$

By virtue of (8.117) the derivative of $v(\lambda, \dot{\lambda})$ can be given (also almost everywhere on $[t_0, \infty)$) as

$$\dot{v} = -A\dot{\lambda}^2 + \frac{c}{\delta}\gamma\gamma_1\lambda\dot{\lambda} + \frac{c}{\delta}\gamma\dot{\lambda}\dot{l} \tag{8.119}$$

where for convenience

$$A = a_2 - (2/c)(1 + \delta\lambda\operatorname{sgn}x) + ck^0/\delta \tag{8.120}$$

With $|l(t)| \leqslant 1$ equation (8.119) becomes

$$\dot{v} = -A\dot{\lambda}^2 \tag{8.121}$$

If, on the other hand, $|l(t)| = 1$ on some interval $[t_1, t_2]$, $t_2 \geqslant t_1$ then

$$\dot{v} = -A\dot{\lambda}^2 + \frac{c}{\delta}\gamma\gamma_1\lambda\dot{\lambda} \tag{8.122}$$

Now we notice that if $|l(t)| = 1$ on $[t_1, t_2]$ then $l(t) = -\operatorname{sgn}\lambda(t)$ for $t \in [t_1, t_2]$, therefore when $\gamma \geqslant \bar{k}^0$ we have on $[t_1, t_2]$ after multiplying (8.116) by $\lambda(t)$:

$$\lambda\dot{\lambda} = (c/\delta)\left[(a_1 - a_2\varkappa + \varkappa^2)\lambda\operatorname{sgn}x - \lambda^2k^0 - \gamma|\lambda|\right],$$

where $\varkappa = (1 + \delta\lambda \operatorname{sgn} x)/c$. From this equation we immediately obtain that for all $t \in [t_1, t_2]$ $\lambda\dot{\lambda} \leqslant 0$. This implies that under the stated conditions for A being a positive constant, the derivative $\dot{v}$ is negative everywhere except the set $\dot{\lambda} = 0$. This set does not contain bounded trajectories, hence by Theorem 5.2 of Ref. 96 the point $(\lambda = 0, \dot{\lambda} = 0)$ is asymptotically stable. Thus we have proved the following lemma.

Lemma 8.5-1. Let all the conditions of Theorems 6.7-1, 8.5-1 and Corollary 6.7-1 be satisfied and

$$\gamma \geqslant \bar{\bar{k}}^0 \tag{8.123}$$

where $\bar{\bar{k}}^0$ is given in (6.72), and

$$k^0 > \frac{1+2\delta}{c}\,\frac{\delta}{c} \tag{8.124}$$

Then for every V_γ solution $(x\,(t),\ \mu\,(t),\ l\,(t))$ of the $S_{\mu\rho\nu}$ system (8.83), (8.109) belonging to the set Γ when $t = t_0$ and for any values of $\mathcal{F}_x$ parameters a_1 and a_2 fixed within (4.2) we have $\lim\limits_{t\to\infty} \mu\,(t) = 0$, $\lim\limits_{t\to\infty} \| x\,(t) \| = 0$, and $\lim\limits_{t\to\infty} l\,(t) = (1/\gamma)\,(a_1 - a_2/c + 1/c^2) \operatorname{sgn} x\,(t_0)$.

The last equality of the theorem immediately follows from (8.116) by letting $\lambda \to 0$. Therefore we only need to comment the inequality (8.124). It is not hard to see that given (6.67) this inequality ensures that the function $A\,(t)$ in (8.120) is positive which in turn warrants that Lyapunov's function $v\,(\lambda, \dot{\lambda})$ in (8.118) is decreasing. The rate at which it decreases is governed thus by $A\,(t)$ dependent on k^0. Choosing this coefficient in an appropriate manner a required damping of $\lambda\,(t)$ may be secured.

In addition, for a sufficiently large k^0, from some time instant we obtain $|\,l\,(t)\,| < 1$, and for all posterior t $\lambda\,(t)$ almost everywhere satisfies the equation $\ddot{\lambda} + A\dot{\lambda} + (c/\delta)\,\gamma\gamma_1\lambda = 0$ and changes sign at most once. With this choice of $S_{\mu\rho\nu}$ system parameters, under the aforestated conditions, the projections of V_γ solutions on the phase plane look as those in Fig. 8.14.

Summarily, we have demonstrated in this section that in the $S_{\mu\nu}$ system augmented by a static OFB loop at sufficiently large t the x^{I} process behavior can be made independent of stepwise variations in $\mathcal{F}_x$ parameters and the function $\lambda\,(t) = \sigma\,(t)/\delta\,|\,x\,(t)\,|$ decaying at a desired rate. The presence of x^{II} processes in such $S_{\mu\rho\nu}$ systems denies, however, the complete solution of the control problem.

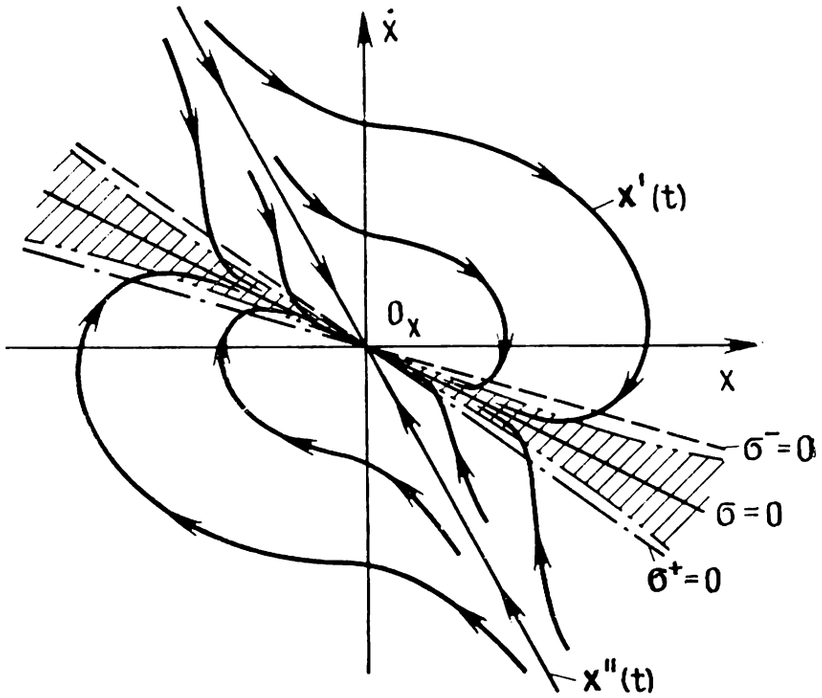

Figure 8.14

Similar conclusions can be drawn for $S_{\mu\rho\nu}$ (2) systems in $\mathcal{P}_x$-process control. In this case the constraint (6.67) on the class of $\mathcal{P}_x$-processes controlled in this way can be lifted and the respective design formulae are simplified substantially.

8.6 $S_{\mu\rho\nu}$ (1) Systems with Inertial Laws of Coordinate-Operator and Operator-Coordinate Feedback

This concluding section studies $S_{\mu\rho\nu}$ (1) systems with inertial COFB and OCFB laws. The rationale behind changing in some cases from integral control laws to inertial laws in binary control systems has been covered in detail in Section 6.8. The present section will handle inertial COFB and OCFB laws corresponding to the block diagram presented in Fig. 8.10. The associated $\mathcal{A}_{\mu\rho\nu}$ (1) control algorithm is as follows:

$$
\begin{aligned}
&\bar{u}\,(t) = u\,(t) + v\,(t)\\
&u\,(t) = \mu\,(t)\,k^0 \mid x\,(t)\mid\\
&v\,(t) = l\,(t)\,\gamma \mid x\,(t)\mid\\
&\dot{\mu}\,(t) = -\alpha\,[\mu\,(t) + \operatorname{sgn}\sigma'\,(t)], \quad t \geqslant t_0\\
&\dot{l}\,(t) = -\gamma_1\,[l\,(t) + \operatorname{sgn}\sigma\,(t)], \quad t \geqslant t_0\\
&\sigma'\,(t) = \sigma\,(t) + \delta\mu\,(t) \mid x\,(t)\mid\\
&\sigma\,(t) = -[c\dot{x}\,(t) + x\,(t)]
\end{aligned}
\qquad (8.125)
$$

Accompanied by the $\mathcal{P}_x$-process equation (8.83) this set describes motions in the $S_{\mu\rho\nu}$ system. Among all solutions $(x(t)\ \mu(t)\ l(t))$ to this system we shall consider only so-called V'_γ solutions for which

$$(x(t_0),\ \mu(t_0),\ l(t_0)) \in V'_\gamma = \{(x,\ \mu,\ l):\ x \in \mathbb{R}^2,\ |\mu(t_0)| < 1,$$

$$|l(t_0)| < 1\}$$

The major properties of V'_γ solutions are outlined in Ref. 93.

A qualitative investigation for this $S_{\mu\rho\nu}$ system can be carried out along the lines numerously described in this book. In short, we establish conditions for the set Γ (7.18) to be attracting and invariant, then the asymptotic behavior of controlled processes is evaluated subject to $V'_\gamma \in \Gamma$, and finally, analysis of asymptotic behavior of $\lambda(t) = \sigma(t)/\delta\ |x(t)|$ is performed. The first two of these problems are easily solvable with the argument of Sections 6.8 and 7.3, therefore here we content ourselves with formulating the principal result and issuing brief comments.

Theorem 8.6-1. Suppose at some positive constants γ, γ_1, c, and δ, $0 < \delta < 1$, and some h, $0 < h < 1$, the condition (6.67) is met and

$$(1 - h)(k^0 + \gamma) = \bar{\bar{k}}^0 \tag{8.126}$$

$$\alpha \geqslant (c/\delta)(\gamma + k^0 + \bar{\bar{k}}^0) \tag{8.127}$$

where $\bar{\bar{k}}^0$ is defined in (6.72). Then the set Γ (7.18) is attracting and invariant on V'_γ solutions $(x(t),\ \mu(t),\ l(t))$ of the $S_{\mu\rho\nu}$ system (8.83) and (8.125). In addition, all controlled processes $x(t)$ divide themselves into two families, $x^{\mathrm{I}}(t)$ and $x^{\mathrm{II}}(t)$ processes. For x^{I} processes there exists a time instant from which $x^{\mathrm{I}}(t) \in G_\delta$ and the bound (6.58) holds, whereas for x^{II} processes $\lim\limits_{t\to\infty} \| x^{\mathrm{II}}(t) \| = 0$ and beginning from a finite time instant $x^{\mathrm{II}}(t) \notin G_\delta$.

We omit the proof of this theorem confining ourselves only to outlining two relationships essential for its proof. One is the differential equation satisfied by $\sigma(t)$ subject to $x(t) \notin G_\delta$, namely,

$$\dot{\sigma} = -\frac{ca_2 - 1}{c}\sigma - \left(\frac{ca_2 - 1}{c} - ca_1\right)x + ck^0\mu\,|x| + c\gamma l\,|x| \tag{8.128}$$

For any number h, $0 < h < 1$, there is a time instant beginning from which $|\mu(t)| > 1 - h$, $|l(t)| > 1 - h$, and both $l(t)$ and $\mu(t)$ are of other sign than $\sigma(t)$. This substantiates the validity of the condition (8.126) and the fact that the set Γ of (7.18) is a domain of attraction.

The second expression describes the function $\sigma'(t)$ for $t \geqslant t_0$ with $\sigma'(t_0) = 0$, namely,

$$\sigma'(t) = \int_{t_0}^{t} (\dot{\sigma} + \delta\mu |\dot{x}| + \delta |x| \dot{\mu})\, d\tau$$
$$= c \int_{t_0}^{t} \left[\left(a_1 - a_2 \frac{\xi_\lambda}{c} + \frac{\xi_\lambda \xi_\mu}{c^2} \right) x \right.$$
$$\left. + k^0 \mu |x| + \gamma l |x| - (\alpha\delta/c) |x| (\mu + \operatorname{sgn} \sigma') \right] d\tau \qquad (8.129)$$

where $\xi_\lambda = 1 + \delta\lambda \operatorname{sgn} x$, and $\xi_\mu = 1 + \delta\mu \operatorname{sgn} x$.

This equation immediately yields that $\sigma'(t) = 0$ for all $t \geqslant t_0$ when (8.127) is met, implying that the set Γ is invariant. For $V_\gamma' \in \Gamma$, $|\sigma(t)| < \delta |x(t)|$ and by Theorem 6.7-1 and Corollary 6.7-1 such controlled processes converge to O_x. Consequently, the $\mathcal{P}_x$-process is $\mathcal{A}_{\mu\rho\nu}$ controlled with the algorithm (8.125). The fact that it is steady $\mathcal{A}_{\mu\rho\nu}$ controlled can be established in the usual manner after devising an analog of Lemma 6.7-2. The result may be formulated as follows: given the condition (6.67) is satisfied

$$\mathcal{P}_x \in \mathcal{C}_{\mathcal{A}_{\mu\rho\nu}(1)}\{\mathcal{F}_c, I_2; N\} \qquad (8.130)$$

where $N = (c_m/\delta)\,[(1 + c + \delta)/c]^2 \exp(2P/\alpha)$, α satisfies (8.127), and P is the same as in (8.112).

Let us now examine the asymptotic behavior of $\lambda(t) = \sigma(t)/\delta |x(t)|$. For $V_\gamma' \in \Gamma$ and $t \geqslant t_0$, this function can be represented as equation (8.126). Assuming again that $\mathcal{P}_x$ parameters a_1 and a_2 are fixed within (4.2), we differentiate this equation with respect to time and upon accounting of $\mu(t) = -\lambda(t)$ and $\dot{l}(t) = -\gamma_1 \times [l(t) + \operatorname{sgn} \lambda(t)]$ obtain that for almost all $t \geqslant t_0$

$$\ddot{\lambda} + [a_2 - (2/c)(1 + \delta\lambda \operatorname{sgn} x) + (c/\delta) k^0] \dot{\lambda}$$
$$+ (c/\delta) \gamma\gamma_1 \operatorname{sgn} \lambda + (c/\delta) \gamma\gamma_1 l = 0 \qquad (8.131)$$

Eliminating $l(t)$ between (8.116) and (8.131) we obtain finally that almost everywhere on the semiaxis $t \geqslant t_0$

$$\ddot{\lambda} + A_1 \dot{\lambda} + (c/\delta) \gamma_1 k^0 \lambda + (c/\delta) \gamma\gamma_1 \operatorname{sgn} \lambda$$
$$- (c/\delta) \gamma_1 k_\lambda = 0 \qquad (8.132)$$

where

$$A_1 = a_2 - (2/c)(1 + \delta\lambda \operatorname{sgn} x) + (c/\delta) k^0 + \gamma_1 \qquad (8.133)$$

$$k_\lambda = \left[a_1 - a_2 \frac{1 + \delta\lambda \operatorname{sgn} x}{c} + \left(\frac{1 + \delta\lambda \operatorname{sgn} x}{c} \right)^2 \right] \operatorname{sgn} x \qquad (8.134)$$

Assuming (8.123) is satisfied, we carry out stability analysis for equation (8.133) with the Lyapunov function

$$v(\lambda, \dot{\lambda}) = \frac{1}{2} \left(\dot{\lambda}^2 + \frac{c}{\delta} \gamma_1 k^0 \lambda^2 \right) + \frac{c}{\delta} \gamma_1 \int_0^\lambda (\gamma \operatorname{sgn} \eta - k_\eta)\, d\eta \qquad (8.135)$$

It is an easy matter to see that for $\gamma \geqslant \bar{\bar{k}}^0$ this function is positive. Its time derivative is

$$\dot{v} = -A_1 \dot{\lambda}^2 \tag{8.136}$$

where A_1 is borrowed from (8.133).

Here again as in Sections 8.4 and 8.5 we see that under the aforestated conditions the point ($\lambda = 0$, $\dot{\lambda} = 0$) is asymptotically stable if: A_1 is a positive constant, $\lim\limits_{t\to\infty} \mu(t) = -\lim\limits_{t\to\infty} \lambda(t) = 0$ and $\lim\limits_{t\to\infty} l(t) = \gamma^{-1} \lim\limits_{\lambda\to 0} k_\lambda$ with k_λ defined in (8.134). We therefore arrived at the following statement.

Lemma 8.6-1. Let all the conditions of Theorem 8.6-1 are met and

$$\gamma \geqslant \bar{\bar{k}}^0 \tag{8.137}$$

$$(c/\delta)\, k^0 + \gamma_1 > (1 + 2\delta)/c \tag{8.138}$$

Then for every V_γ' solution ($x(t)$, $\mu(t)$, $l(t)$) of the $S_{\mu\rho\nu}$ system (8.83), (8.125) such that at $t = t_0$ $V_\gamma' \in \Gamma$, and for any $\mathcal{P}_x$ parameters a_1 and a_2 fixed within (4.2), $\lim\limits_{t\to\infty} \mu(t) = 0$, $\lim\limits_{t\to\infty} \|x(t)\| = 0$, and $\lim\limits_{t\to\infty} l(t) = \gamma^{-1}(a_1 - a_2/c + 1/c^2)\ \mathrm{sgn}\ x(t_0)$.

Comparing equation (8.132) with equation (8.117) we see that as compared with the integral control laws the use of inertial laws in COFB and OCFB loops simplifies control system structure and provides additional potentialities for affecting asymptotic behavior of $\lambda(t)$. For example, its decay can be made aperiodic and consistent with a given exponential bound.

In other words Lemma 8.6-1 presents the conditions to make the dynamic behavior of x^1 processes independent of specific $\mathcal{P}_x$ parameters at infinite times.

To conclude this section we note that all the findings and conclusions of this section apply also to $\mathcal{A}_{\mu\rho\nu}$ (2) algorithms or to S system control. The respective relationships for control system parameters are easily derivable with the general scheme of reasoning developed in this treatise.

Conclusion

A great deal of today's control problems concern essentially nonstationary and ill-defined dynamic systems operating under parametrical uncertainty. Traditional design techniques frequently fail to ensure the attainment of strict enough control objectives. Moreover, the choice of control has to be made under the ever increasing pressure of various constraints imposed on gain constants, informat-

ion, complexity of the mathematical model, etc. Therefore, a control theory is utterly desirable which would account for modern plant structure and the associated practical constraints. Such a theory should obviously rely on essentially nonlinear control algorithms. System design methods should therefore evolve capable of identifying the general configuration of algorithms and control systems in simple terms comprehensible to a control engineer.

The theory of binary control systems is a solver for some of the aforementioned problems. More specifically, the aim of this study was to demonstrate that essentially nonlinear control systems designed with the classical feedback principle can be a useful approach for handling a number of applications operating under formidable constraints. The results of this study inspire a great deal of optimism concerning binary control systems. These are, of course, but the first steps towards efficient and adaptable automatic systems guided by some principles of algorithm generation rather than by one, though sophisticated control algorithm. A control system conceptualized in this way forms the control algorithms proper on-line in response to the situation at hand. Instead of simple primary loop algorithms, binary control systems employ complicated nonlinear control algorithms in this feedback loop with simple algorithms operable in other control loops forming a hierarchical superstructure for the primary loop.

For more involved control problems or a lower amount of information on the controlled process, the system acquires ever more hierarchical loops with simple control laws, providing together a complicated nonlinear solver. This hierarchical buildup seems to reflect some general concepts of sophisticated control algorithm construction deserving a separate study.

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INDEX

A

Algorithm
- $\mathcal{A}_k$, 41
- $\mathcal{A}_\mu(q)$, 55
- $\mathcal{A}_{\mu\rho}$, 160
- $\mathcal{A}^T_{\mu\rho}$, 170
- quasicontinuous control, 192
- $\mathcal{A}_{\mu\rho\nu}$, 223, 247, 264
- $\mathcal{A}_{\mu\nu}$, 253

Asymptotic independence, 57

B

Binary dynamic systems, 16
Binary principle, 14, 16

C

Control
- Δ-accurate, 90
- ill-defined $\mathcal{P}_x$-process, 56

Control systems
- adaptive, 12
- feedback, 20

Coordinate, 11, 15
Controlled process, 137

E

Error-closing control principle, 20

F

Feedback
- bang-bang CO, 75, 226
- coordinate, 17, 51
- coordinate-operator, 18, 55
- coordinate-parametric, 29
- dynamic OC, 230
- inertial CO, 110, 124
- integral CO, 89, 118
- new types of, 17
- operator, 18, 154
- operator-coordinate, 29
- proportional coordinate operator, 61, 227
- time invariant, linear coordinate, 41

G

G^0_T invariance, 178

I

Inaccuracies in model approximation, 53

L

Linear zone, 49

O

Open-loop control principle, 20
Operator
- coordinate-control, 14
- variant, 16
- variable-parameter, 12

Operator-signal, 14
Operator transformation, 16
Operator variable, 14

P

$\mathcal{P}$-parameters, 41

S

Systems
- binary control, structures of, 28
- closed-loop dynamic, 20
- exponentially $\mathcal{A}$-controlled, 40
- ill-defined dynamic, 23
- $S_\mu(q)$, 55
- $S_{\mu\rho}$, 160, 165, 189, 206
- $S_{\mu\rho\nu}$, 223
- $S_{\mu\nu}$, 236
- VS_h, 115
- VS (2), 209
- with distributed parameters, 10
- with lumped parameters, 10

T

Transformation
- coordinate, 16
- coordinate-operator, 16
- operator-coordinate, 16

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